The classical KMS condition for Hamiltonian PDEs

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Outline:

- Bose-Hubbard model on (finite) graphs
  - Dynamical mean field limit
  - Variational mean field limit
- Equilibrium states
  - Gibbs states, Gibbs measures
  - High temperature regime
- High temperature limit for Bose-Hubbard
  - KMS states, KMS measures
  - Convergence result
- KMS condition for Hamiltonian PDEs
  - Hamiltonian PDEs
  - Gaussian measures: infinite dimension
  - Results and conjectures

The discrete Laplacian

- A finite graph $\mathcal{G} = (V, \mathcal{E})$ where V is the set of vertices and $\mathcal{E}$ is the set of edges.
- Assume that $\mathcal{G}$ is a simple undirected graph.
- The degree of each vertex $x \in V$ is denoted by $\deg(x)$.
- We denote the graph by $\mathcal{G}$ or V.

The Hilbert space of all complex-valued functions on V denoted as $\ell^2(\mathcal{G})$ and endowed with its natural scalar product and with the orthonormal basis $(e_x)_{x \in V}$ such that

$$e_x(y) := \delta_{x,y}, \quad \forall x, y \in V.$$

Definition

The discrete Laplacian on the graph $\mathcal{G}$ is a non-positive bounded operator on $\ell^2(\mathcal{G})$ given by,

$$(\Delta_{\mathcal{G}}\psi)(x) := -deg(x)\psi(x) + \sum_{y \in V, y \sim x} \psi(y),$$

with the above sum running over the nearest neighbors of x and ψ is any function in $\ell^2(\mathcal{G})$.

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Fock space and CCR's relations

- Consider the bosonic Fock space,

$$\mathfrak{F} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \otimes_s^n \ell^2(\mathcal{G}) \ ,$$

where $\otimes_s^n \ell^2(\mathcal{G})$ denotes the symmetric n-fold tensor product of $\ell^2(\mathcal{G})$.

- Introduce the usual creation and annihilation operators acting on the bosonic Fock space,

$$a_x = a(e_x) \quad \text{and} \quad a_x^* = a^*(e_x) \, ,$$

then the following canonical commutation relations are satisfied,

$$\left[a_x, a_y^*\right] = \delta_{x,y} 1_{\mathfrak{F}} \quad \text{and} \quad \left[a_x^*, a_y^*\right] = [a_x, a_y] = 0, \quad \forall x, y \in V \, .$$

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The Bose-Hubbard Hamiltonian

Definition (Bose-Hubbard Hamiltonian)

For $\varepsilon \in (0, \bar{\varepsilon})$, $\lambda > 0$ and $\kappa < 0$, define the ε-dependent Bose-Hubbard Hamiltonian on the bosonic Fock space $\mathfrak{F}$ by

$$H_\varepsilon^{\mathrm{BH}} := \frac{\varepsilon}{2} \sum_{x,y\in V: y\sim x} (a_x^* - a_y^*)(a_x - a_y) + \frac{\varepsilon^2 \lambda}{2} \sum_{x\in V} a_x^* a_x^* a_x a_x - \varepsilon\kappa \sum_{x\in V} a_x^* a_x .$$

Here λ is the on-site interaction, κ is the chemical potential and ε is the semiclassical parameter.

Remark

The first term of the Hamiltonian $H_\varepsilon^{\mathrm{BH}}$ is the kinetic part of the system and corresponds to the second quantization of the discrete Laplacian. Indeed, one can write

$$\frac{1}{2} \sum_{x,y\in V: y\sim x} (a_x^* - a_y^*)(a_x - a_y) = \sum_{x\in V} deg(x)\, a_x^* a_x - \sum_{x,y\in V, y\sim x} a_x^* a_y = \mathrm{d}\Gamma(-\Delta_{\mathcal{G}}) .$$

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Dynamical mean field limit

For N number of particles, define the many-body Hamiltonian H_N :

$$H_N \equiv \varepsilon^{-1} H_\varepsilon^{\mathrm{BH}}\Big|_{\otimes_s^N \ell^2(\mathcal{G})} \quad \text{with } \varepsilon = N^{-1}$$

(Mean-field regime for H_N) $N \to \infty \quad \equiv \quad \varepsilon \to 0$ (Semiclassical regime for $H_\varepsilon^{\mathrm{BH}}$)

Theorem (PhD C. Rouffort)

The mean-field regime of the Bose-Hubbard Hamiltonian is governed by the Discrete nonlinear Schrödinger equation (DNLS) given by

$$\begin{cases} i\partial_t u_n(t) & = -\big((\Delta_{\mathcal{G}} + \kappa)u(t)\big)_n + \frac{\lambda}{2}|u_n(t)|^2 u_n(t), \quad \forall n \in V \\ u(t=0) & = u(0) \in \ell^2(\mathcal{G}) \end{cases} \tag{1}$$

Mean-field propagation:

- Coherent states
- Factorized states
- Wigner measures

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Variational mean field limit

- Ground state energy:

$$E_N := \inf_{\|\psi^N\|_{\otimes_s^N \ell^2(\mathcal{G})}=1} \langle \psi^N, H_N \psi^N \rangle_{\mathfrak{F}} = \frac{1}{N} \inf_{\|\psi^N\|_{\otimes_s^N \ell^2(\mathcal{G})}=1} \langle \psi^N, H^{\mathrm{BH}}_{N^{-1}} \psi^N \rangle_{\mathfrak{F}}.$$

- The DNLS Hamiltonian energy:

$$h(u) = -\langle u, (\Delta_{\mathcal{G}} + \kappa) u \rangle_{\ell^2(\mathcal{G})} + \frac{\lambda}{4} \sum_{n \in V} |u_n|^4. \qquad (2)$$

Theorem (PhD Q. Liard)

In the mean field regime the ground state energy of H_N is well described by the lowest level energy of the Discrete nonlinear Schrödinger equation (DNLS):

$$\lim_{N \to \infty} E_N = \inf_{\|u\|_{\ell^2(\mathcal{G})}=1} h(u) < \infty,$$

$$\text{where} \qquad H_N \equiv \varepsilon^{-1} H^{\mathrm{BH}}_{\varepsilon}\Big|_{\otimes_s^N \ell^2(\mathcal{G})} \qquad \text{with } \varepsilon = N^{-1}.$$

Bose-Hubbard at positive temperature

- So far the discussion on Bose-Hubbard was about the zero temperature.
- The Bose-Hubbard Hamiltonian defines a $\mathscr{W}^*-$ dynamical system $(\mathfrak{M}, \alpha_t)$ where $\mathfrak{M}$ is the von Neumann algebra of all bounded operators $\mathscr{B}(\mathfrak{F})$ on the Fock space and α_t is the one parameter group of automorphisms defined by

$$\alpha_t(A) = e^{i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{BH}}}\, A\, e^{-i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{BH}}} \,,$$

  for any $A \in \mathfrak{M}$.
- Question: Can we describe the equilibrium states of the Bose-Hubbard model at positive temperature in the semiclassical regime?

Bose-Hubbard equilibrium states

- Here we consider a finite graph.

Lemma (Partition function)

Since the chemical potential $\kappa < 0$ then

$$\mathrm{Tr}_{\mathfrak{F}}\left(e^{-\beta H_{\varepsilon}^{\mathsf{BH}}}\right) < \infty. \tag{3}$$

Definition (Gibbs equilibrium states)

The Gibbs equilibrium state of the Bose-Hubbard $\mathscr{W}^*-$ dynamical system $(\mathfrak{M}, \alpha_t)$ at inverse temperature $\varepsilon\beta$ is well defined, according to (3), and it is given by

$$\omega_{\varepsilon}(A) = \frac{\mathrm{Tr}_{\mathfrak{F}}(e^{-\beta H_{\varepsilon}^{\mathsf{BH}}} A)}{\mathrm{Tr}_{\mathfrak{F}}(e^{-\beta H_{\varepsilon}^{\mathsf{BH}}})}\,. \tag{4}$$

DNLS Gibbs measures

Denote

$$E = \ell^2(\mathcal{G}) \equiv \mathbb{R}^{2d}.$$

For a negative chemical potential $\kappa < 0$,

$$z_\beta = \int_E e^{-\beta h(u)}\, dL < +\infty\,, \tag{5}$$

for all $\beta > 0$. Here dL is the Lebesgue measure on E.

Definition (DNLS Gibbs measure)

The equilibrium Gibbs measure of the DNLS Hamiltonian system (1) at inverse temperature $\beta > 0$, is the Borel probability measure given by

$$\mu_\beta = \frac{e^{-\beta h(\cdot)}\, dL}{\int_E e^{-\beta h(u)}\, dL} \equiv \frac{1}{z_\beta}\, e^{-\beta h(\cdot)}\, dL\,. \tag{6}$$

High temperature limit

The high temperature regime corresponds to

$$(\text{Temperature}) \quad T_\varepsilon = \frac{1}{\beta_\varepsilon} \to \infty \quad \equiv \quad \beta_\varepsilon = \varepsilon\beta \to 0 \quad (\text{Inverse temperature}).$$

Hence

Hight temperature regime $\equiv$ Semiclassical regime $\equiv$ Mean field regime

Claim

In the high temperature limit the Bose-Hubbard Gibbs state ω_ε converges towards the DNLS Gibbs measure μ_β.

- Entropy and Berezin-Lieb inequality
- Dyson-Schwinger expansion
- Kubo-Martin-Schwinger (KMS) condition ?

Quantum KMS states

- An element $A \in \mathfrak{M}$ is entire analytic if and only if for any $t > 0$ the sum below is convergent,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|S^n(A)\| < \infty \,, \quad \text{where} \quad S(\cdot) = \frac{i}{\varepsilon}[H_\varepsilon^{\mathrm{BH}}, \cdot]. \tag{7}$$

- We denote the set of entire analytic elements by $\mathfrak{M}_\alpha$.
- The dynamics α_t can be extended to complex times through the following absolutely convergent sum,

$$\alpha_z(A) = \sum_{n=0}^{\infty} \frac{z^n}{n!} S^n(A) \,, \quad \forall z \in \mathbb{C} \,.$$

Definition (Quantum KMS states)

We say that a state ω is a $(\alpha_t, \varepsilon\beta)$-KMS state if and only if ω is trace-class (normal) and for any $A, B \in \mathfrak{M}_\alpha$,

$$\omega(A\, \alpha_{i\varepsilon\beta}(B)) = \omega(BA) \,. \tag{8}$$

Classical KMS measures

- The classical Kubo-Martin-Schwinger (KMS) condition was introduced by Gallavotti and Verboven in the seventies.
- The Poisson bracket is given by

$$\{F, G\} := \frac{1}{i}\,\left(\partial_u F \cdot \partial_{\bar{u}} G - \partial_u G \cdot \partial_{\bar{u}} F\right)\,. \tag{9}$$

Definition (Classical KMS measures)

A Borel probability measure μ on $\ell^2(\mathcal{G})$ is a classical KMS measure for the DNLS equation at inverse temperature β if for any F, G smooth functions on $\ell^2(\mathcal{G})$,

$$\beta\,\int_E \{h, G\}\,(u)\;F(u)\,d\mu(u) = \int_E \{F, G\}\,(u)\,d\mu(u)\,. \tag{10}$$

Here h is the classical Hamiltonian of the DNLS equation given by (2).

Characterization of equilibrium

- Characterization of Quantum equilibrium:

Proposition (Quantum KMS)

The Bose-Hubbard Gibbs state ω_ε, i.e.

$$\omega_\varepsilon(A) = \frac{\mathrm{Tr}_{\mathfrak{F}}(e^{-\beta H_\varepsilon^{\mathrm{BH}}} A)}{\mathrm{Tr}_{\mathfrak{F}}(e^{-\beta H_\varepsilon^{\mathrm{BH}}})},$$

is the unique KMS state of the $\mathscr{W}^*-$ dynamical system $(\mathfrak{M}, \alpha_t)$ at the inverse temperature $\varepsilon\beta$.

- Characterization of Classical equilibrium:

Proposition (Classical KMS, Am.-RATSIMANETRIMANANA)

The DNLS Gibbs measure $\mu_\beta = \frac{e^{-\beta\, h(u)}}{z(\beta)} dL$, $z(\beta) = \int_E e^{-\beta\, h(u)}\, dL(u)$, is the unique classical KMS measure of the DNLS equation at inverse temperature β.

Convergence of KMS conditions

- A Borel probability measure μ on $\ell^2(\mathcal{G})$ is a Wigner measure of $\{\omega_\varepsilon\}_{\varepsilon\in(0,\bar{\varepsilon})}$ if there exists a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\varepsilon_k = 0$ and for any $f \in \ell^2(\mathcal{G})$,

$$\lim_{k\to\infty} \underbrace{\omega_{\varepsilon_k}(W(f))}_{\text{Generating functional}} = \lim_{k\to\infty} \omega_{\varepsilon_k}\left(e^{i\sqrt{\varepsilon_k}\,\Phi(f)}\right) = \underbrace{\int_{\ell^2(\mathcal{G})} e^{i\sqrt{2}\Re\mathrm{e}\langle f,u\rangle}\, d\mu}_{\text{Characteristic function}}. \tag{11}$$

Theorem (Am.-RATSIMANETRIMANANA)

Let ω_ε be the KMS state of the Bose-Hubbard $\mathscr{W}^*-$ dynamical system $(\mathfrak{M},\alpha_t)$ at inverse temperature $\beta_\varepsilon = \varepsilon\,\beta$ given by (4). Then any Wigner measure μ of the family $\{\omega_\varepsilon\}_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies the classical KMS condition, i.e.: for any F, G smooth functions on $E = \ell^2(\mathcal{G})$,

$$\beta \int_E \{h, G\}(u)\, F(u)\, d\mu(u) = \int_E \{F, G\}(u)\, d\mu(u)\,. \tag{12}$$

where h is the DNLS classical Hamiltonian given by (2) and $\{\cdot,\cdot\}$ denotes the Poisson bracket recalled in (9).

Outline

Hamiltonian PDEs: Linear system

Consider a positive operator $A : D(A) \subseteq H \to H$ such that,

$$\exists c > 0, \quad A \geq c\mathbb{1} \,. \tag{13}$$

A Hamiltonian dynamical system is given by the quadratic Hamiltonian function,

$$h_0 : D(A^{1/2}) \to \mathbb{R}, \qquad h_0(u) = \frac{1}{2}\langle u, Au\rangle \,. \tag{14}$$

In this case, the vector field is a linear operator $X_0 : D(A) \to H$,

$$X_0(u) = -iAu,$$

and the linear field equation governing the dynamics of the system is:

$$\dot{u}(t) = X_0(u(t)) = -iAu(t) \,. \tag{15}$$

Hamiltonian PDEs: Compactness condition

We suppose that the operator A admits a compact resolvent.

- There exists an orthonormal basis in H of eigenvectors $\{e_j\}_{j\in\mathbb{N}}$ of A with their eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$,

$$A e_j = \lambda_j \, e_j \, . \tag{16}$$

- Furthermore, assume the following assumption:

$$\exists s \geq 0 : \quad \sum_{j=1}^{\infty} \frac{1}{\lambda_j^{1+s}} < +\infty. \tag{17}$$

We note that $\{e_j, ie_j\}_{j\in\mathbb{N}}$ is an O.N.B of $H_{\mathbb{R}}$.

Hamiltonian PDEs: Weighted Sobolev spaces

Weighted Sobolev spaces w.r.t the operator A are constructed as follows: For any $r \in \mathbb{R}$, define the inner product:

$$\forall x, y \in \mathcal{D}(A^{\frac{r}{2}}) , \qquad \langle x, y \rangle_{H^r} := \langle A^{r/2} x, A^{r/2} y \rangle .$$

- H^s denotes the Hilbert space $(\mathcal{D}(A^{s/2}), \langle \cdot, \cdot \rangle_{H^s})$ with $s \geq 0$.
- H^{-s} denotes the completion of the pre-Hilbert space $(\mathcal{D}(A^{-s/2}), \langle \cdot, \cdot \rangle_{H^{-s}})$.
- Hilbert rigging: One has the canonical continuous and dense embedding,

$$H^s \subseteq H \subseteq H^{-s} .$$

We note that H^{-s} identifies also with the dual space of H^s relatively to the inner product of H.

Gaussian measures: infinite dimension

The free Gibbs measure written formally as

$$\mu_{\beta,0} \equiv \frac{e^{-\beta h_0(\cdot)}\, du}{\int e^{-\beta h_0(u)}\, du},$$

is rigorously defined as a Gaussian measure on the Hilbert space H^{-s}.

Definition (Gaussian measure)

▷ The mean-vector of $\mu \in \mathcal{P}(H^{-s})$ is the vector $m \in H^{-s}$ such that:

$$\langle f, m \rangle_{H^{-s}_{\mathbb{R}}} = \int_{H^{-s}} \langle f, u \rangle_{H^{-s}_{\mathbb{R}}}\, d\mu, \qquad \forall f \in H^{-s}.$$

If $m = 0$, one says that μ is a centered measure.

▷ The covariance operator of $\mu \in \mathcal{P}(H^{-s})$ is a linear operator $Q : H^{-s}_{\mathbb{R}} \to H^{-s}_{\mathbb{R}}$ such that:

$$\langle f, Q\, g \rangle_{H^{-s}_{\mathbb{R}}} = \int_{H^{-s}} \langle f, u - m \rangle_{H^{-s}_{\mathbb{R}}} \langle u - m, g \rangle_{H^{-s}_{\mathbb{R}}}\, d\mu, \quad \forall f, g \in H^{-s}.$$

▷ $\mu \in \mathcal{P}(H^{-s})$ is Gaussian if $B \mapsto \mu(\{y \in H^{-s} : \langle x, y \rangle_{H^{-s}_{\mathbb{R}}} \in B\})$ are Gaussian measures on $\mathbb{R}$.

i.e: Centred Gaussian measures are Gibbs measures over infinite dimensional spaces.

Hamiltonian PDE: Nonlinear system

Nonlinear Hamiltonian system:

- Linear operator A satisfying the compactness condition in (13) and (16).
- Nonlinear functional $h^I : H^{-s} \to \mathbb{R}$ satisfying for some $\beta > 0$:

$$\forall p \in [1, \infty) \qquad e^{-\beta h^I(\cdot)} \in L^p(\mu_{\beta,0}) \qquad \text{and} \qquad h^I \in \underbrace{\mathbb{D}^{1,p}(\mu_{\beta,0})}_{\text{Gross-Sobolev spaces}} \tag{18}$$

- The Hamiltonian function:

$$h(u) = \frac{1}{2}\langle u, Au\rangle + h^I(u) = h_0(u) + h^I(u)\,. \tag{19}$$

The vector field of the system is

$$X(u) = -iAu - i\nabla h^I(u) = X_0(u) + X^I(u)\,, \tag{20}$$

defines a (non-autonomous) field equation in the interaction representation:

$$\dot{u}(t) = e^{itA} X^I(e^{-itA} u(t))\,.$$

Gibbs measures: Gross-Sobolev spaces

Lemma (Malliavin derivative)

The following linear operator is closable:

$$\begin{aligned} \nabla : \mathscr{C}^\infty_{c,cyl}(H^{-s}) \subset L^p(\mu_{\beta,0}) &\longrightarrow L^p(\mu_{\beta,0}; H^{-s}), \\ F = \varphi \circ \pi_n &\longmapsto \nabla F = \sum_{j=1}^{2n} \partial_j \varphi(\pi_n(\cdot))\, f_j . \end{aligned}$$

The Malliavin derivative is the closure of such linear operator (still denoted by ∇).

Definition (Gross-Sobolev spaces)

The Gross-Sobolev space $\mathbb{D}^{1,p}(\mu_{\beta,0})$ is the closure domain of the Malliavin derivative ∇ with respect to the norm:

$$\|F\|^p_{\mathbb{D}^{1,p}(\mu_{\beta,0})} := \|F\|^p_{L^p(\mu_{\beta,0})} + \|\nabla F\|^p_{L^p(\mu_{\beta,0};H^{-s})} . \tag{21}$$

Gibbs measures: infinite dimension

Nonlinear Hamiltonian system:

- Linear operator A satisfying the compactness condition.
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The vector field of the system is

$$X(u) = -iAu - i\nabla h^I(u) = X_0(u) + X^I(u) \,. \tag{23}$$

Definition (Gibbs measure)

The Gibbs measure of the dynamical system (23), at inverse temperature $\beta > 0$, is:

$$\mu_\beta = \frac{e^{-\beta h^I(\cdot)}\, d\mu_{\beta,0}}{\int_{H^{-s}} e^{-\beta h^I(u)}\, d\mu_{\beta,0}} \equiv \frac{1}{z_\beta}\, e^{-\beta h^I(\cdot)}\, d\mu_{\beta,0} \,. \tag{24}$$

Classical KMS condition: Hamiltonian PDE

The classical Kubo-Martin-Schwinger (KMS) condition, introduced by Gallavotti and Verboven, characterizes the Gibbs measures.

Definition (Classical KMS states)

A measure $\mu \in \mathcal{P}(H^{-s})$ is a classical KMS state, at inverse temperature β, for the Hamiltonian system (26)-(27) if and only if for all $F, G \in \mathscr{C}^{\infty}_{c,cyl}(H^{-s})$,

$$\int_{H^{-s}} \{F, G\}(u)\, d\mu = \beta \int_{H^{-s}} \langle \nabla F(u), X(u) \rangle\, G(u)\, d\mu\,, \tag{25}$$

with the Poisson bracket $\{\cdot, \cdot\}$ defined in (33).

Here the Hamiltonian function is

$$h(u) = \frac{1}{2}\langle u, Au \rangle + h^I(u) = h_0(u) + h^I(u)\,, \tag{26}$$

and the vector field of the system is

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Classical KMS condition: Hamiltonian PDE

Assume for all $p \in [1, \infty)$:

$$e^{-\beta h^I(\cdot)} \in L^p(\mu_{\beta,0}) \qquad \text{and} \qquad h^I \in \mathbb{D}^{1,p}(\mu_{\beta,0})\,. \tag{28}$$

Theorem (Am.-Sohinger)

Let $\mu \in \mathcal{P}(H^{-s})$ such that $\mu \ll \mu_{\beta,0}$ and suppose that

$$\frac{d\mu}{d\mu_{\beta,0}} \in \mathbb{D}^{1,2}(\mu_{\beta,0})\,.$$

Then μ is a classical KMS measure of the PDE Hamiltonian system (26)-(27) if and only if μ is equal to the Gibbs measure, i.e.:

$$\mu_\beta = \frac{e^{-\beta h^I}\ \mu_{\beta,0}}{\int_{H^{-s}} e^{-\beta h^I(u)} d\mu_{\beta,0}} = \mu\,.$$

Conjecture

Claim

The Quantum KMS condition for Quantum field Hamiltonians converges in the hight temperature limit towards the Classical KMS condition of corresponding Hamiltonian PDEs ?

References

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- Hight temperature limit:
  - Mathieu Lewin, Phan Thành Nam, and Nicolas Rougerie
  - Jürg Fröhlich, Antti Knowles, Benjamin Schlein, and Vedran Sohinger
  - Andrew Rout and Vedran Sohinger
- Wigner measures: Am.-Francis Nier, Sébastien Breteaux, Michele Correggi, Marco Falconi, Shahnaz Farhat, Marco Olivieri, and recently Benjamin Alvarez

Cylindrical smooth functions

Definition

Let $\{f_j\}_{j\in\mathbb{N}}$ O.N.B of $H_{\mathbb{R}}$. Consider for $n\in\mathbb{N}$ the mapping $\pi_n : H^{-s}\to\mathbb{R}^{2n}$,

$$\pi_n(x) = \left(\langle x, f_1\rangle_{H_{\mathbb{R}}}, \dots, \langle x, f_{2n}\rangle_{H_{\mathbb{R}}}\right). \tag{29}$$

Define $\mathscr{C}^{\infty}_{c,cyl}(H^{-s})$ as the set of all functions $F : H^{-s}\to\mathbb{R}$ such that

$$F = \varphi\circ\pi_n \tag{30}$$

for some $n\in\mathbb{N}$ and $\varphi\in\mathscr{C}^{\infty}_{c}(\mathbb{R}^{2n})$.

The gradient of F at the point $u\in H^{-s}$ is

$$\nabla F(u) = \sum_{j=1}^{2n}\partial_j\varphi(\pi_n(u))\, f_j\,, \tag{31}$$

where $\partial_j\varphi$ are the partial derivatives with respect the $2n$ coordinates of φ.

Poisson structure

Consider:

- The algebra of smooth bounded cylindrical functions $\mathscr{C}^{\infty}_{b,cyl}(H^{-s})$.
- $F, G \in \mathscr{C}^{\infty}_{b,cyl}(H^{-s})$ such that: $\forall u \in H^{-s}$,

$$F(u) = \varphi \circ \pi_n(u) \,, \qquad G(u) = \psi \circ \pi_m(u) \,, \tag{32}$$

with $\varphi \in \mathscr{C}^{\infty}_{b}(\mathbb{R}^{2n})$ and $\psi \in \mathscr{C}^{\infty}_{b}(\mathbb{R}^{2m})$ for some $n, m \in \mathbb{N}$.

Definition

Then, for all such $F, G \in \mathscr{C}^{\infty}_{b,cyl}(H^{-s})$, the Poisson bracket is:

$$\{F, G\}(u) := \sum_{j=1}^{\min(n,m)} \partial_j^{(1)}\varphi(\pi_n(u))\, \partial_j^{(2)}\psi(\pi_m(u)) - \partial_j^{(1)}\psi(\pi_m(u))\, \partial_j^{(2)}\varphi(\pi_n(u)) \,. \tag{33}$$

Convergence argument

- The Bose-Hubbard $(\alpha_t, \varepsilon\beta)$-KMS state ω_ε (formally) satisfies:

$$\omega_\varepsilon \left(W(f)\, \alpha_{i\varepsilon\beta}(W(g)) \right) = \omega_\varepsilon \left(W(g)\, W(f) \right) .$$

- A simple computation leads to:

$$\omega_\varepsilon \left(W(f)\, \frac{\alpha_{i\varepsilon\beta}(W(g)) - W(g)}{i\varepsilon} \right) = \omega_\varepsilon \left(\frac{[W(g), W(f)]}{i\varepsilon} \right) . \tag{34}$$

- Key arguments:

$$\lim_{k\to\infty} \omega_{\varepsilon_k} \left(\frac{[W(g), W(f)]}{i\varepsilon_k} \right) = \int_E \{ e^{\sqrt{2} i \Re\mathrm{e} \langle g, u\rangle}, e^{\sqrt{2} i \Re\mathrm{e} \langle f, u\rangle} \} \, d\mu , \tag{35}$$

$$\lim_{k\to\infty} \omega_{\varepsilon_k} \left(W(f)\, \frac{\alpha_{i\varepsilon\beta}(W(g)) - W(g)}{i\varepsilon} \right) = \beta \int_E \{ e^{\sqrt{2} i \Re\mathrm{e} \langle g, u\rangle}, h(u) \} \, e^{\sqrt{2} i \Re\mathrm{e} \langle f, u\rangle} \, d\mu . \tag{36}$$

Remark

No Dyson expansion nor Variational (entropy) arguments are used.