High frequency uniform resolvent estimates for the magnetic Laplacian

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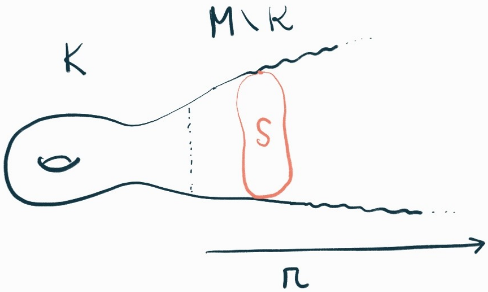
Effective Approximation and Dynamics of Many-Body Quantum Systems

Manifold (M, G) $M = K \cup (M \setminus K)$ with $M \setminus K$ infinite end and dimension $n \geq 3$.

We make assumptions on the asymptotic behaviour of the metric on $M \setminus K$.



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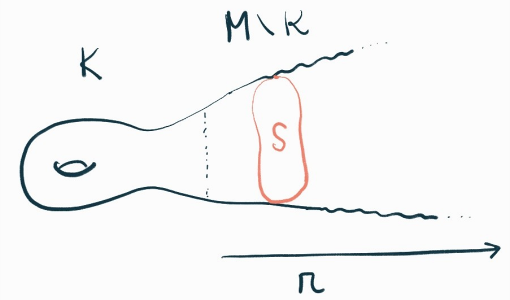

Example: Writing $x = r\omega$ with $r = |x|$, $\omega = \frac{x}{|x|}$ we can represent $\mathbb{R}^n_*$ with

$$\mathbb{R}^n_* \rightsquigarrow ((0, \infty) \times \mathbb{S}^{n-1}, dr^2 + r^2 d\sigma)$$

and $\mathbb{R}^n$ outside of a compact obstacle with $((R, \infty) \times \mathbb{S}^{n-1}, dr^2 + r^2 d\sigma)$.

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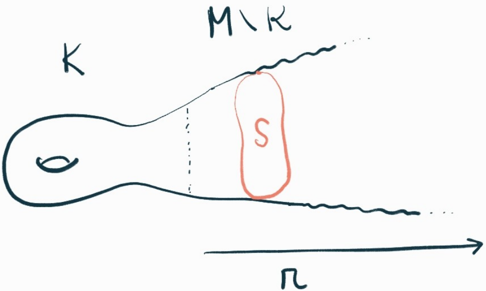

More generally, instead of the sphere we can consider $(S, g(r))$ where $g(r)$ is perturbation of a fixed metric $\overline{g}$ on S.

$$M \setminus K \rightsquigarrow ((R, \infty) \times S, dr^2 + r^2 g(r)).$$

We say then that M is asymptotically conical .

Introduction

Remark: We can actually write the metric in a more general form

$$dr^2 + l(r)^{-2} g(r)$$

where

$$O(1) \geq -\frac{l'(r)}{l(r)} \geq \frac{c}{r} \qquad c > 0 \quad (\text{for ex. } l(r) = r^{-1},\ l(r) = e^{-r}).$$

This allows to include the asymptotically conical and asymptotically hyperbolic cases at the same time.

On M we can consider the Laplace-Beltrami operator P_0. The operator is selfadjoint, hence

$$(P_0 - z)^{-1}$$

is well defined for any $z \in \mathbb{C} \setminus \mathbb{R}^+$.

We are interested in the study of the boundary value of the resolvent

$$(P_0 - \lambda^2 + i0)^{-1} := \lim_{\varepsilon \to 0} (P_0 - \lambda^2 + i\varepsilon)^{-1}$$

that is when the spectral parameter $z = \lambda^2 - i\varepsilon$ approaches the resolvent set.

We want to obtain uniform estimates when $\varepsilon \to 0$.

Example: On $\mathbb{R}^3$ we can write the integral kernel of the resolvent, to bound the resolvent we would need to evaluate the L^2 norm of

$$(-\Delta - z^2)^{-1} f(x) = \int \frac{e^{iz|x-y|}}{4\pi|x-y|} f(y) dy = \int e^{-Imz|x-y|} \frac{e^{iRez|x-y|}}{4\pi|x-y|} f(y) dy$$

this produces a bound in $|Imz|^{-1}$ when regarded as an operator on L^2.

More in general, we recall that

$$\|(P_0 - z)^{-1}\|_{L^2 \to L^2} \leq \frac{1}{|Imz|}$$

so there is no hope to obtain uniform bounds in ε just in L^2.

The right operator topology to consider is the one of operators on weighted L^2 spaces

$$L^2_s = L^2(\langle x \rangle^s dx), \;\; s > 1/2.$$

We can study $(P_0 - \lambda^2 + i0)^{-1}$ as an operator of $\mathcal{B}(L^2_s, L^2_{-s})$, which translates on L^2 bounds on

$$\langle x \rangle^{-s} (P_0 - \lambda^2 + i0)^{-1} \langle x \rangle^{-s}.$$

- Writing functional calculus via the spectral measure and then using Stone's formula

$$e^{itP_0} = \int e^{it\lambda}\, dE_{P_0}(\lambda)$$

Why is the limit $\varepsilon \to 0$ interesting?

- Writing functional calculus via the spectral measure and then using Stone's formula

$$e^{itP_0} = \int e^{it\lambda}\, dE_{P_0}(\lambda) = \int e^{it\lambda} Im(P_0 - \lambda^2 + i0)^{-1}\, d\lambda$$

if we have an L^2 bound on $Im(P_0 - \lambda^2 + i0)^{-1}$ in $\mathcal{B}(L^2_s, L^2_{-s})$ we can obtain time decay estimates on $\langle x\rangle^{-s} e^{itP_0} \langle x\rangle^{-s}$.

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$$\|\langle x\rangle^{-s}(P_0 + V - \lambda^2 \pm i0)^{-1}\|_{L^2 \to L^2}.$$

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- Asymptotic expansion of the spectral shift function holds if $(P_0 - \lambda^2 \pm i0)^{-1}$ has bound $O(e^{c\lambda})$ in the $\mathcal{B}(L^2_s, L^2_{-s})$ topology (Robert '94).

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  (Idea: integrate $(P_0 - z)^{-1}$ on an exponentially small stripe in a contour deformation argument.)

Question

We will be interested in a perturbation of the Laplace-Beltrami operator P_0.

To define P_0 on M we use the quadratic form

$$\langle D_G u, D_G v\rangle_{L^2(M)}$$

where D_G is the gradient induced by the metric on M. We modify this quadratic form by taking $D_G - A$ with A a decaying vector field.

We then obtain the magnetic Laplacian P on (M, G) via the quadratic form

$$\langle (D_G - A)u, (D_G - A)v\rangle_{L^2(M)}.$$

In P we can also include a decaying potential (long range).

We will then consider

$$(P - \lambda^2 + i\varepsilon)^{-1}$$

when $\varepsilon \to 0$ in the appropriate weighted spaces.

References for the Laplacian with a potential

On $\mathbb{R}^n$:

- When no trapped geodesics: bound in $O(\lambda^{-1})$ with weights $\langle x \rangle^{-1/2-\mu}$ (Robert-Tamura '88, Wang '88, Gerard Martinez '89)
- When some trapping is allowed: outside of an obstacle (Ikawa '85)

On non compact manifolds:

- At worse: $O(e^{c\lambda})$ (Burq '98 outside of an obstacle, Burq '02 long range potential)
- At best: $O(\lambda^{-1})$ (Cardoso-Vodev '02)
- With some trapping: $O(\lambda^{-1/2}\log\lambda)$ (Nonnenmacher-Zworski '09, Datchev '09) $O(\lambda^{-\sigma}), \ \sigma < 1$ (Christianson-Wunsch '13)

Goals

- Obtain estimates in the case of the perturbed operator P.
- Optimise the weights in considering decaying weights of strenght $-1/2$ in Besov-like spaces (as suggested from results on $\mathbb{R}^n$, Agmon-Hörmander '76).

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- Have a control for all high frequencies $\lambda \gg 1$ to which other approaches, like Mourre theory, are not well adapted if no assumption is added.

Example: The main assumption of Mourre theory is a positive commutator estimate. On $\mathbb{R}^n$ one exploits the property

$$i[-\Delta, \mathcal{A}] = 2(-\Delta)$$

where $\mathcal{A}$ is the generator of dilations $\mathcal{A} = \frac{x\cdot\nabla+\nabla\cdot x}{2i}$. For more general operators one can hope to obtain for the suitable $\mathcal{A}$

$$\phi(P)i[P, \mathcal{A}]\phi(P) \geq \phi^2(P)P + \phi(P)K\phi(P).$$

The remainder $\phi(P)K\phi(P)$ can be treated with additional assumptions.

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Perspectives:

- Possibly push forward the frequency independent approach of the first part of the work to obtain estimates for all intermediate and high frequencies.

One of the goals is improving the weights.

We recall that at infinity the manifold is of the form

$$((R, +\infty) \times S, dr^2 + r^2 g(r)),$$

the unbounded direction is the radial one, hence the weight will be $r^{-1/2}$.

In the region $(R, +\infty) \times S$

1. we partition $(R, +\infty)$ in dyadic intervals $[2^{k-1}, 2^{k+1}]$,
2. on $[2^{k-1}, 2^{k+1}]$ we consider $\|r^{1/2} v\|_{L^2(drdg, 2^{k-1} \leq r \leq 2^{k+1})}$
3. take the ℓ^1 norm of the sequence at step 2.

This defines

$$\|v\|_{B_{>R}} = \sum_{k \geq k_0} \|r^{1/2} v\|_{L^2(drdg, 2^{k-1} \leq r \leq 2^{k+1})}.$$

For the dual norm we replace

$$r^{1/2} \rightsquigarrow r^{-1/2} \qquad \ell^1 \rightsquigarrow \ell^\infty$$

and obtain

$$\|v\|_{B^*_{>R}} = \sup_{k \geq k_0} \|r^{-1/2} v\|_{L^2(drdg, 2^{k-1} \leq r \leq 2^{k+1})}.$$

Result

Theorem (V.G. '23)

Let $u \in H^2(M)$, $\lambda \gg 1$,

$$X_R := (R, +\infty) \times S$$

and $R_1 > R$. There exists a constant $C > 0$ independent of λ and ε sucht that

$$\|u\|_{H^1(M \setminus X_R)} + \|u\|_{H^1, B^*_{>R}} \leq O(\lambda^{-1} e^{\lambda C}) \|(P - \lambda^2 + i\varepsilon)u\|_{L^2(M \setminus X_{R_1})} + O(\lambda^{-1} e^{\lambda C}) \|(P - \lambda^2 + i\varepsilon)u\|_{B_{>R}},$$

where $\|u\|_{H^1, B^*_{>R}}$ includes the norms of u and of its derivatives.

Comments:

- A bound on $\|u\|_{H^1, B^*_{>R}}$ implies the more common bound on L^2 weighted spaces with weight $r^{-1/2-\mu}$.
- Unlike Vodev ('02) and Cardoso, Vodev ('02) we include perturbations of order one.
- The first part of our proof is not restricted to the high frequency regime .
- We do not allow cusps.

To be more precise, what we can prove is

$$\|u\|_{L^2(M\setminus X_{R_1})} + O(e^{\lambda C})\|u\|_{B^*_{>R}} \leq O(\lambda^{-1}e^{\lambda C})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})} + O(\lambda^{-1}e^{\lambda C})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}},$$

where $M \setminus X_{R_1}$ is a bounded region. For a function supported at infinity we can simplify the exponential factors.

Hence if we cut away from the compact portion of M we obtain a bound in $O(\lambda^{-1})$.

Corollary (Cutoff resolvent V.G. '23)

Let $\lambda \gg 1$. There exists $R_1 > R$ and χ a smooth cutoff on $X_{R_1} = (R_1, +\infty) \times S$ such that for all $\varepsilon > 0$

$$\|\chi(P-\lambda^2+i\varepsilon)^{-1}\chi\|_{B_{>R_1}\to B^*_{>R_1}} \leq O(\lambda^{-1})$$

in particular

$$\|r^{-1/2-\mu}\chi(P-\lambda^2+i\varepsilon)^{-1}\chi r^{-1/2-\mu}\|_{L^2\to L^2} \leq O(\lambda^{-1}).$$

We can directly see that the $B_{>R_1} \to B^*_{>R_1}$ bounds imply the $L^2_{1/2+\mu} \to L^2_{-1/2-\mu}$ ones.

We can just use the inclusions

$$L^2_{1/2+\mu}(r > R_1) \hookrightarrow B_{>R_1}, \quad B^*_{>R_1} \hookrightarrow L^2_{-1/2-\mu}(r > R_1).$$

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$$\begin{aligned} \|r^{-1/2-\mu} f\|_{L^2(drdg, r>R_1)} &\leq \sum_{k \geq k_1} 2^{\mu(1-k)} \|r^{-1/2} f\|_{L^2(drdg, 2^{k-1} \leq r \leq 2^{k+1})} \\ &\leq \sup_{k \geq k_1} \|r^{-1/2} f\|_{L^2(drdg, 2^{k-1} \leq r \leq 2^{k+1})} \sum_{k \geq k_1} 2^{\mu(1-k)} \leq O(1) \|f\|_{B^*_{>R_1}}. \end{aligned}$$

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$$\|r^{-1/2-\mu} f\|_{L^2(drdg, r>R_1)} \leq \sum_{k \geq k_1} 2^{\mu(1-k)} \|r^{-1/2} f\|_{L^2(drdg, 2^{k-1} \leq r \leq 2^{k+1})}$$

$$\leq \sup_{k \geq k_1} \|r^{-1/2} f\|_{L^2(drdg, 2^{k-1} \leq r \leq 2^{k+1})} \sum_{k \geq k_1} 2^{\mu(1-k)} \leq O(1) \|f\|_{B^*_{>R_1}}.$$

Strategy

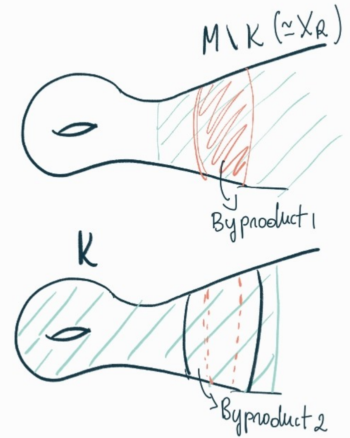
1. On $M \setminus K$ bound u by $(P - \lambda^2 + i\varepsilon)u$ up to a small term.

   Byproduct: the H^1 norm of u over a compact set (because we have no semiclassical parameter)

2. On K use Carleman estimates to bound u by $(P - \lambda^2 + i\varepsilon)u$.

   Byproduct: the H^1 norm of u over a compact set. The term is of exponential size in λ.

3. We bound the exponentially large remainders thanks to a λ-dependent weight.



Remark: Only in step 3 we need to use the assumption $\lambda \gg 1$.

Step 1: Estimates on the infinite end (Any frequency $\lambda > \lambda_0$)

In this part we consider the region

$$((R, +\infty) \times S, dr^2 + r^2 g(r)).$$

We find that for any $\delta \in (0, 1)$ there exists $c(\delta) > 1$ and a compact subset $K(\delta)$ of $(R, +\infty) \times S$ such that

$$\|u\|_{H^1, B^*_{>R}} \leq O(\delta^{-1}\lambda^{-1})\|(P - \lambda^2 + i\varepsilon)u\|_{B_{>R}} + O(\delta)\|u\|_{H^1, B^*_{>R}} + c(\delta)\|u\|_{H^1(K(\delta))}$$

for all $u \in H^2(M)$.

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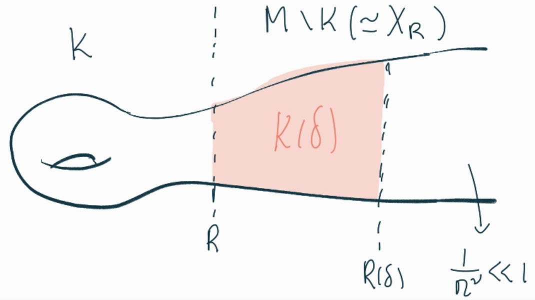
Byproduct of step 1.

Remark: The additional term on $K(\delta)$ is necessary to get the the absorbable term $O(\delta)$.

Why? λ^{-1} is not necessarily small ($\lambda > \lambda_0$ only). In a way, we have no semiclassical parameter available that can give us the smallness we want in δ.

Fix: We can recover smallness thanks to decaying factors $r^{-\nu}$ with $\nu > 0$ provided r is large enough.



In other words, the norm for $r \geq R(\delta)$ contributes to the term $O(\delta)$ and the norm for $R \leq r \leq R(\delta)$ is the contribution of $K(\delta)$.

Step 2: Estimates on bounded region (Any frequency $\lambda > \lambda_0$)

From

$$\|u\|^2_{H^1,B^*_{>R}} \leq O(\delta^{-1}\lambda^{-2})\|(P-\lambda^2+i\varepsilon)u\|^2_{B_{>R}} + O(\delta)\|u\|^2_{H^1,B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}$$

we fix $\delta_0 \in (0,1)$ such that $O(\delta_0) < 1$ and after moving $O(\delta)\|u\|^2_{H^1,B^*_{>R}}$ to the left we have

$$\|u\|_{H^1,B^*_{>R}} \leq O(\lambda^{-1})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}} + O(1)\|u\|_{H^1(K(\delta_0))},$$

This bounds the norm of u on $X_R = (R,+\infty)\times S$.

For the remaining compact region we use a consequence of Carleman estimates due to Lebeau and Robbiano ('95). This gives the exponential factors $O(e^{\lambda C})$.

Proposition (Interpolation inequality G.L.-L.R. '95, G.L.-L.R.-J.L '22)

Let $\mathcal{M}$ a Riemannian manifold, $\mathcal{T}$ the Laplace-Berltrami operator on $\mathcal{M}$ and $\mathcal{R}$ a differential operator of order one. Let σ, W, V open subsets of $\mathcal{M}$ such that

$$\sigma \subset W \subset V, \quad \overline{W} \cap \partial\mathcal{M} = \emptyset.$$

There exist $c > 0, \gamma \in (0,1)$ such that

$$\|v\|_{H^1(W)} \leq c\|v\|_{H^1(V)}^{1-\gamma}(\|(\mathcal{T} + \mathcal{R} - \partial_t^2)v\|_{L^2(V)} + \|v\|_{L^2(\sigma)})^\gamma$$

for any $v \in H^2(\mathcal{M})$.

If we apply it to $\mathcal{M} = (-1,1)_t \times \mathcal{M}_0$ and $v = v(t,m) = e^{tz}u(m), m \in \mathcal{M}_0$

$$(\mathcal{T} - \partial_t^2 + \mathcal{R})v = e^{tz}(\mathcal{T} + \mathcal{R} - z^2)u(m)$$

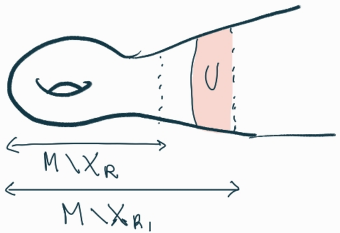
which we will use with $\mathcal{T} + \mathcal{R} = P$ and $z^2 = \lambda^2 - i\varepsilon$.

We recall that $M \setminus X_R$ is a compact region. From step 2 we obtain

$$\|u\|_{H^1(M\setminus X_R)} \leq O(e^{\lambda/\gamma})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})} + O(e^{\lambda/\gamma})\|u\|_{H^1(U)}$$

with $R_1 > R$, U open bounded subset of X_R and $\gamma \in (0,1)$.

The right hand side is of the same form of the one in step 1.



Remark: we can choose U such that $K(\delta_0) \subset U$ so that the remainders are contained in the same compact region.

Step 3: Combining spatial infinity and the bounded region.

(Frequencies $\lambda \gg 1$)

Up to now we have

$$\|u\|_{H^1,B^*_{>R}} \leq O(\lambda^{-1})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}} + O(1)\|u\|_{H^1(K(\delta_0))},$$

$$\|u\|_{H^1(M\setminus X_R)} \leq O(e^{\lambda/\gamma})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})} + O(e^{\lambda/\gamma})\|u\|_{H^1(U)}.$$

with $K(\delta_0)\subset U$. Adding the two inequalities

$$\begin{aligned}\|u\|_{H^1(M\setminus X_R)} + \|u\|_{H^1,B^*_{>R}} \leq & O(\lambda^{-1})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}} \\ & + O(e^{\lambda/\gamma})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})} + O(e^{\lambda/\gamma})\|u\|_{H^1(U)}\end{aligned}$$

for all $\lambda > \lambda_0$.

Let $\varphi(r)$ a smooth increasing function such that $\varphi > 1/\gamma$ on U then

$$\|e^{\lambda/\gamma}u\|_{H^1(U)} = \|e^{\lambda(1/\gamma-\varphi)}e^{\lambda\varphi}u\|_{H^1(U)} \leq \underbrace{O(e^{-\lambda c})}_{<1}\|e^{\lambda\varphi}u\|_{H^1(U)}$$

becomes a small term, since $\lambda \gg 1$. This implies that in its bound we are allowed some remainder term, since we will be able to absorb them anyways.

Let U' a bounded region with $U \subset U'$, we bound $\|e^{\lambda\varphi}u\|_{H^1(U')}$ by

$$O(e^{-\lambda c})\|e^{\lambda\varphi}u\|_{H^1(U')} \leq O(\lambda^{-1}e^{\lambda C})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})} + O(\lambda^{-1}e^{\lambda C})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}} + O(e^{-\lambda c})\|u\|_{H^1(U'')}$$

with U'' a bounded region in $M \setminus X_R$. We use this to conclude

$$\|u\|_{H^1(M\setminus X_R)} + \|u\|_{H^1,B^*_{>R}} \leq O(\lambda^{-1})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}} + O(e^{\lambda/\gamma})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})} + O(e^{\lambda/\gamma})\|u\|_{H^1(U)}$$

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where the norm on U'' is absorbable by $M \setminus X_R$. So finally

$$\begin{aligned}\|u\|_{H^1(M\setminus X_R)} + \|u\|_{H^1,B^*_{>R}} \leq & O(\lambda^{-1}e^{\lambda C})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}} \\ & + O(\lambda^{-1}e^{\lambda C})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})}.\end{aligned}$$

We recall $K(\delta_0) \subset U \subset U'$, so trivially the norm on $K(\delta_0)$ can be bounded by the one on U'. With more careful computations we find

$$e^{\lambda\varphi_*}\|u\|_{H^1(K(\delta_0))} \leq O(e^{-\lambda c})\|e^{\lambda\varphi}u\|_{H^1(U')}$$

thanks to the properties of φ. Then we can bound

$$\begin{aligned} e^{\lambda\varphi_*}\|u\|_{H^1,B^*_{>R}} \leq& e^{\lambda\varphi_*}O(\lambda^{-1})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}} + O(e^{-\lambda c})\|e^{\lambda\varphi}u\|_{H^1(U')} \\ \leq& e^{\lambda\varphi_*}O(\lambda^{-1})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}} \\ &+ O(\lambda^{-1}e^{\lambda C})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})} + O(e^{-\lambda c})\|u\|_{H^1(U'')}. \end{aligned}$$

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We combine the two regions by adding together $\|u\|_{L^2(M\setminus X_{R_1})}$ and $e^{\lambda\varphi_*}\|u\|_{B^*_{>R}}$ to find

$$\begin{aligned} \|u\|_{L^2(M\setminus X_{R_1})} + e^{\lambda\varphi_*}\|u\|_{B^*_{>R}} \leq & e^{\lambda\varphi_*}O(\lambda^{-1})\|(P-\lambda^2+i\varepsilon)u\|_{B_{>R}} \\ & + O(\lambda^{-1}e^{\lambda C})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})} \\ & + O(e^{-\lambda c})\|u\|_{H^1(U'')} \end{aligned}$$

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If u is supported at infinity away from $M \setminus X_{R_1}$ we obtain the bound $O(\lambda^{-1})$ on the cutoff resolvent.

Some computations

In step 1 we had for any $\delta \in (0,1)$

$$\|u\|^2_{H^1,B^*_{>R}} \leq O(\delta^{-1}\lambda^{-2})\|(P-\lambda^2+i\varepsilon)u\|^2_{B_{>R}} + O(\delta)\|u\|^2_{B^*_{>R}} + c(\delta)\|u\|^2_{H^1(K(\delta))}.$$

Assuming we have bounds on the norms of the radial and angular derivatives of u, we will show how to obtain $\|u\|_{B^*_{>R}}$. We recall

$$\|u\|_{B^*_{>R}} = \sup_{k\geq k_0} \|r^{-1/2}u\|_{L^2(drdg, 2^{k-1}\leq r\leq 2^{k+1})},$$

hence we need to bound $\|r^{-1/2}u\|_{L^2(drdg,2^{k-1}\leq r\leq 2^{k+1})}$ uniformly in k. We consider

$$h = \lambda^{-1} \quad h^2P - 1 + i\varepsilon'.$$

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Let χ_k a cutoff on $(2^{k-1},2^{k+1})$ then

$$Re(\chi_k^2 u,(h^2P-1+i\varepsilon')u) = -\|\chi_k u\|^2_{L^2} + Re(\chi_k^2 u, h^2Pu)$$

implies

$$\|r^{1/2}r^{-1/2}\chi_k u\|^2_{L^2}$$

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implies

$$2^{k-1}\|r^{-1/2}u\|^2_{L^2(drdg,2^{k-1}\leq r\leq 2^{k+1})} \leq \|r^{1/2}r^{-1/2}\chi_k u\|^2_{L^2}$$

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Let χ_k a cutoff on $(2^{k-1},2^{k+1})$ then

$$Re(\chi_k^2 u,(h^2P-1+i\varepsilon')u) = -\|\chi_k u\|^2_{L^2} + Re(\chi_k^2 u, h^2Pu)$$

implies

$$\begin{aligned}2^{k-1}\|r^{-1/2}u\|^2_{L^2(drdg,2^{k-1}\leq r\leq 2^{k+1})} &\leq \|\chi_k u\|^2_{L^2}\\ &= -Re(\chi_k u,\chi_k(h^2P-1+i\varepsilon')u) + Re(\chi_k u,\chi_k h^2 Pu)\end{aligned}$$

In step 1 we had for any $\delta \in (0,1)$

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$$2^{k-1}\|r^{-1/2}u\|^2_{L^2(drdg, 2^{k-1}\leq r\leq 2^{k+1})} \leq \|\chi_k u\|^2_{L^2}$$
$$\leq -Re(r^{-1/2}\chi_k u, r^{1/2}\chi_k(h^2P - 1 + i\varepsilon')u) + 2^{k+1}Re(\chi_k r^{-1/2}u, \chi_k r^{-1/2}h^2Pu)$$

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implies

$$2^{k-1}\|r^{-1/2}u\|^2_{L^2(drdg,2^{k-1}\leq r\leq 2^{k+1})} \leq \|\chi_k u\|^2_{L^2}$$
$$\leq -Re(r^{-1/2}\chi_k u, r^{1/2}\chi_k(h^2P-1+i\varepsilon')u) + 2^{k+1}Re(\chi_k r^{-1/2}u, \chi_k r^{-1/2}h^2Pu)$$

Now by Cauchy-Schwartz if $\delta \in (0,1)$

$$
\begin{aligned}
|Re(r^{-1/2}\chi_k u, r^{1/2}\chi_k(h^2P-1+i\varepsilon')u)| \leq\ & O(\delta)\|r^{-1/2}u\|^2_{L^2(drdg, 2^{k-1}\leq r\leq 2^{k+1})} \\
& + O(\delta^{-1})\|r^{1/2}(h^2P-1+i\varepsilon')u\|^2_{L^2(drdg, 2^{k-1}\leq r\leq 2^{k+1})} \\
\leq\ & O(\delta)\|r^{-1/2}u\|^2_{L^2(drdg, 2^{k-1}\leq r\leq 2^{k+1})} \\
& + O(\delta^{-1})\|(h^2P-1+i\varepsilon')u\|^2_{B_{>R}},
\end{aligned}
$$

where we have the desired quantity $(h^2P-1+i\varepsilon')u$ and an absorbable term .

We still have

$$
\begin{aligned}
& \|r^{-1/2}u\|^2_{L^2(drdg, 2^{k-1}\leq r\leq 2^{k+1})} \\
& \quad \leq Re(r^{-1/2}\chi_k u, r^{1/2}\chi_k(h^2P-1+i\varepsilon')u) + Re(\chi_k r^{-1/2}u, \chi_k r^{-1/2}h^2Pu).
\end{aligned}
$$

In the term $Re(\chi_k r^{-1/2}u, \chi_k r^{-1/2}h^2Pu)$ we will exchange decay for smallness which generated the extra term $\|u\|_{H^1(K(\delta))}$.

We recall P is the magnetic Laplacian, it contains for example a radially decaying potential

$$V_m.$$

If we consider the contribution of V_m

$$\begin{aligned}|Re(\chi_k r^{-1/2}u, \chi_k r^{-1/2}V_m u)| =& |Re(\chi_k r^{-1/2}u, r^{-\nu}(r^{\nu}V_m)\chi_k r^{-1/2}u)| \\ \leq& O(2^{\nu(1-k)})\|r^{-1/2}u\|^2_{L^2(drdg, 2^{k-1}\leq r\leq 2^{k+1})} \\ \leq& O(2^{\nu(1-k)})\|u\|^2_{B^*_{>R}}\end{aligned}$$

which is an absorbable term if $2^{\nu(1-k)}$ is small enough. This is true for $k \gg 1$, or in other words on an interval $(2^{k-1}, 2^{k+1})$ which is far away at radial infinity.

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We know how to bound

$$|Re(\chi_k r^{-1/2}u, \chi_k r^{-1/2}V_m u)| \quad \text{for } k \geq \kappa(\delta)$$

which leaves us with all the previous intervals $(2^{k-1}, 2^{k+1})$, $k_0 \leq k < \kappa(\delta)$ that make up our compactly supported term $\|u\|_{H^1(K(\delta))}$.

It remains to consider the contributions in

$$Re(\chi_k r^{-1/2} u, \chi_k r^{-1/2} h^2 P u)$$

given by the differential terms of P.

The idea is that P is of order two and "$P^{1/2}$" defines the H^1 norm.

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The idea is that P is of order two and "$P^{1/2}$" defines the H^1 norm.

We recall that we are in the region $(R, +\infty) \times S$ where S is the angular manifold. For example, in $h^2 P$ the term

$$M(r) \geq 0 \quad \text{selfadjoint}$$

is the part of the operator in the angular variables. Thanks to the selfadjointness (and since $M(r)$ acts only on the radial variables)

$$\begin{aligned} |Re(\chi_k r^{-1/2} u, \chi_k r^{-1/2} M(r) u)| =& |Re(\chi_k r^{-1/2} M(r)^{1/2} u, \chi_k r^{-1/2} M(r)^{1/2} u)| \\ =& \|r^{-1/2} M(r)^{1/2} u\|^2_{L^2(drdg, 2^{k-1} \leq r \leq 2^{k+1})} \end{aligned}$$

which is part of the norm $\| \cdot \|_{H^1, B^*_{>R}}$.

For $\|r^{-1/2} M(r)^{1/2} u\|_{L^2(drdg, 2^{k-1} \leq r \leq 2^{k+1})}$ we use the bound on the angular derivatives $\|M(r)^{1/2} u\|_{B^*_{>R}}$.

Step 2. is an application of the inequality

$$\|v\|_{H^1(V_0)} \leq O(e^{|z|/\gamma})(\|(\mathcal{T}+\mathcal{R}-z^2)v\|_{L^2(V_0)} + \|v\|_{L^2(\sigma_0)}) \quad \sigma_0 \subset V_0,\ V_0 \cap \partial M_0 = \emptyset.$$

to

$$\mathcal{T}+\mathcal{R} = P \quad z^2 = \lambda^2 - i\varepsilon.$$

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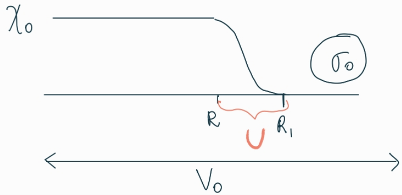
to

$$\mathcal{T}+\mathcal{R} = P \quad z^2 = \lambda^2 - i\varepsilon.$$

To get rid of the control term $\|v\|_{L^2(\sigma_0)}$ we apply it to $\chi_0 u$ with $\chi_0 \equiv 0$ on σ_0.



Then

$$\begin{aligned}\|u\|_{H^1(M\setminus X_R)} &\leq O(e^{\lambda/\gamma})\|(P-\lambda^2+i\varepsilon)(\chi_0 u)\|_{L^2(M\setminus X_{R_1})} \\ &\leq O(e^{\lambda/\gamma})\|(P-\lambda^2+i\varepsilon)u\|_{L^2(M\setminus X_{R_1})} + O(e^{\lambda/\gamma})\underbrace{\|[P,\chi_0]u\|_{L^2(M\setminus X_{R_1})}}_{\leq \|u\|_{H^1(U)}}\end{aligned}$$

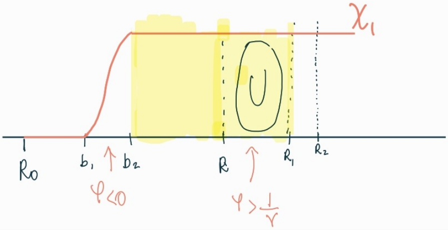
In step 3. we use the fact that φ is negative somewhere . We need to prove a bound on a region containing U, so say $\|e^{\lambda\varphi}u\|_{H^1(X_{b_2}\setminus X_{R_1})}$.

To do so, we use an inequality which holds for functions vanishing on R_0.

We can apply it to $\chi_1 u$ and the right hand side yields terms of the form

$$O(\lambda^{-1})\|e^{\lambda\varphi}(P-\lambda^2+i\varepsilon)(\chi_1 u)\|_{L^2(X_{R_0}\setminus X_{R_2})}$$
$$\leq O(\lambda^{-1})\|e^{\lambda\varphi}(P-\lambda^2+i\varepsilon)u\|_{L^2(X_{R_0}\setminus X_{R_2})}$$
$$+O(\lambda^{-1})\|e^{\lambda\varphi}u\|_{H^1(X_{b_1}\setminus X_{b_2})}$$

where on $X_{b_1}\setminus X_{b_2}$ the factor $e^{\lambda\varphi}$ provides a small parameter since $\varphi<0$ in this region and $\lambda\gg 1$.



This means that we are able to absorb $O(\lambda^{-1})\|e^{\lambda\varphi}u\|_{H^1(X_{b_1}\setminus X_{b_2})}$ on the left by $\|u\|_{H^1(M\setminus X_R)}$.