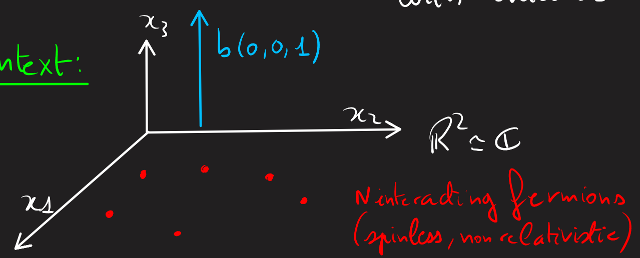
Semi-classical limit of the 2D Hartree equation in a large magnetic field

with Nicolas Rougerie

Context:



question: dynamic when $N, b \to +\infty$?

Goal: Hartree $\to$ Gyrokinetic transport

motivation: QHE

Model:

magnetic Laplacian: $\mathcal{L}_b := (i\hbar\nabla + bA)^2 = \sum_{n\in\mathbb{N}} 2\hbar b\left(n+\frac{1}{2}\right)\Pi_n$ — n^{th} Landau level projection

Symmetric gauge: $A := \frac{X^\perp}{2} = \frac{1}{2}\begin{pmatrix}-x_2\\ x_1\end{pmatrix}$, $\nabla\wedge A = (0,0,1)$, magnetic lenght: $\ell_b := \sqrt{\frac{\hbar}{b}}$

position operator in $\mathbb{R}^2$

Hartree equation: $i\hbar\partial_t\gamma = [\mathcal{L}_b + V + w * \rho_\gamma, \gamma]$ where $\gamma \in \mathfrak{L}^1(L^2(\mathbb{R}^2))$, $\mathrm{Tr}[\gamma] = 1$, $\gamma \geq 0$ (density matrix)

and $V : \mathbb{R}^2 \to \mathbb{R}$, $w : \mathbb{R}^2 \to \mathbb{R}$; $\rho_\gamma(z) := \gamma(z,z)$

scaling: $\hbar \to 0$, $b \to +\infty$ s.t. $\hbar b \to 1$

Newton: $z'' = F + bz'^\perp \Rightarrow z(t) = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\cos(bt)\\ \sin(bt)\end{pmatrix} + \frac{F^\perp}{b}t$ — motion on time scale $\mathcal{O}(b)$

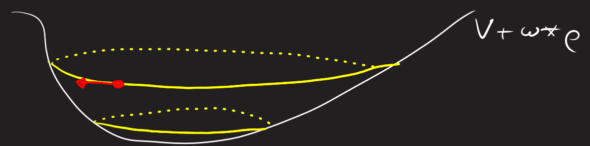
$\gamma_b(t) := \gamma(bt)$ then $\partial_t\gamma_b = \frac{1}{i\ell_b^2}[\mathcal{L}_b + V + w * \rho_{\gamma_b}, \gamma_b]$ (H)

Pauli principle: $\gamma_b \leq 2\pi\ell_b^2$, surfacic degeneracy of a LL: $1/2\pi\ell_b^2$

ex: $\gamma_b := \frac{1}{N}\sum_{i=1}^{N}|u_i\rangle\langle u_i|$ with $(u_i)_i \perp$, $N = \frac{1}{2\pi\ell_b^2}$ so $\mathrm{Tr}[\gamma_b] = 1$, $0 \leq \gamma_b \leq \frac{1}{N} = 2\pi\ell_b^2$, say γ_b is a FDM

$\mathcal{O}(1)$ volume

Gyrokinetic transport equation: $\partial_t\rho + \nabla^\perp(V + w*\rho)\cdot\nabla\rho = 0$
with $\rho : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$



Results:

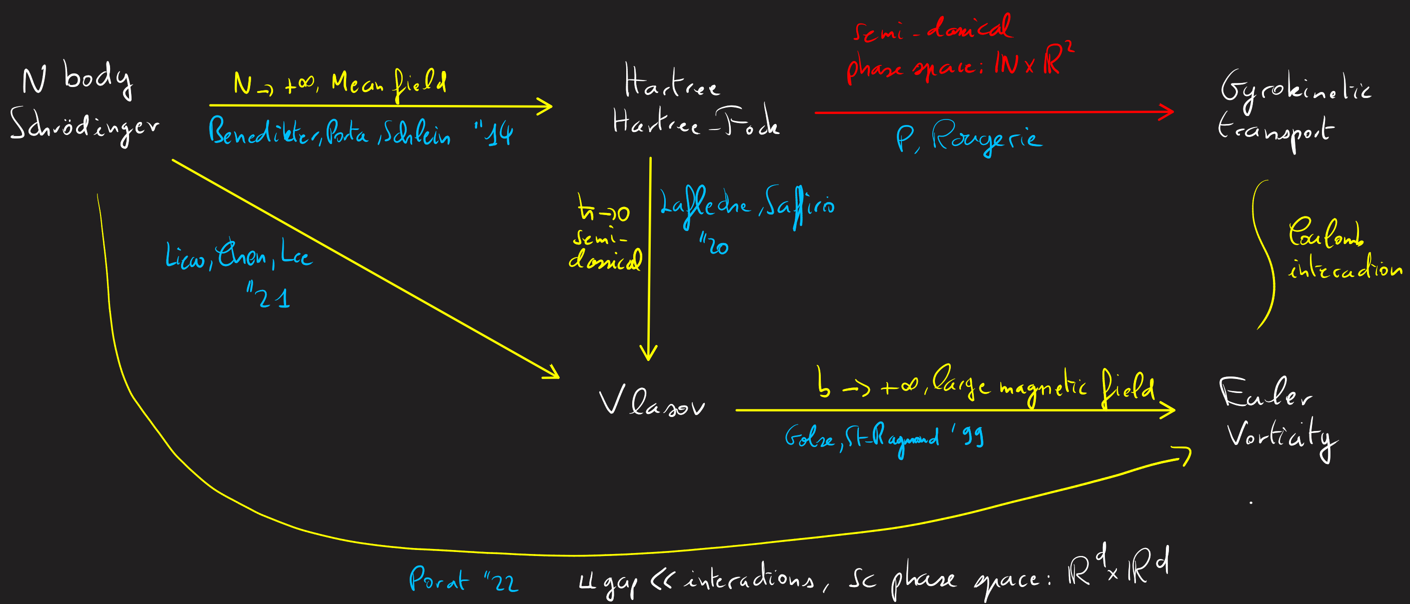
Theorem: Convergence of densities

Let γ_b solves (H), assume $\gamma_b(0)$ is a FDM, $\mathrm{Tr}\left[\gamma_b(0)\left(\mathcal{L}_b + V + \frac{1}{2}w * \rho_{\gamma_b(0)}\right)\right] \leq C$, $V, w \in W^{4,\infty}(\mathbb{R}^2)$

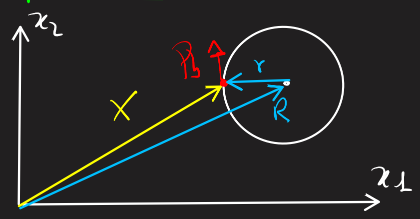
then, up to a subsequence $\begin{cases}\rho_{\gamma_b} \xrightarrow[b\to+\infty]{*} \rho \in \mathcal{M}(\mathbb{R}_+\times\mathbb{R}^2)\\ \rho_{\gamma_b} \xrightarrow[b\to+\infty]{*} \rho_0 \in \mathcal{M}(\mathbb{R}^2)\end{cases}$ and $\forall \varphi \in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^2)$,

$$\int_{\mathbb{R}_+\times\mathbb{R}^2}\rho\left(\partial_t\varphi + \nabla^\perp(V + w*\rho)\cdot\nabla\varphi\right) - \int_{\mathbb{R}^2}\varphi(0)\rho_0\,dz = 0$$

Effective theories graph:



quantization:



operators:	position	anihilation	creation
cyclotron movement	$r := \frac{P_b^\perp}{b}$	$a := \frac{r_2 - i r_1}{\sqrt{2}\hbar_b}$	$a^+ := \frac{r_2 + i r_1}{\sqrt{2}\hbar_b}$
orbit center	$R := X - r$	$\hat{b} := \frac{R_1 - iR_2}{\sqrt{2}\hbar_b}$	$\hat{b}^+ := \frac{R_1 + iR_2}{\sqrt{2}\hbar_b}$

prop: $\mathcal{L}_b$ diagonalization

$[a,a^+] = [\hat{b},\hat{b}^+] = \mathrm{Id}$, $[a,\hat{b}] = [a,\hat{b}^+] = 0$ solves $a\varphi_{0,0} = 0, \hat{b}\varphi_{0,0} = 0$

$\varphi_{n,m} := \frac{(a^+)^n(\hat{b}^+)^m}{\sqrt{n!m!}}\varphi_{0,0}$ with $\varphi_{0,0}(x) := \frac{1}{\sqrt{2\pi}\hbar_b}e^{-\frac{|x|^2}{4\hbar_b^2}}$ is a basis of $L^2(\mathbb{R}^2)$ and $\Pi_n = \sum_{m\in\mathbb{N}}|\varphi_{n,m}\rangle\langle\varphi_{n,m}|$

coherent state: Let $z\in\mathbb{C}$, $\varphi_{n,z} := e^{\frac{\bar{z}\hat{b}^+ - z\hat{b}}{\sqrt{2}\hbar_b}}\varphi_{n,0} = e^{-\frac{|z|^2}{4\hbar_b^2}}\sum_{m\in\mathbb{N}}\frac{\left(\frac{\bar{z}}{\sqrt{2}\hbar_b}\right)^m}{\sqrt{m!}}\varphi_{n,m}$ then $\bar{R}\varphi_{n,z} = \bar{z}\varphi_{n,z}$

$\Pi_z := \sum_{n\in\mathbb{N}}|\varphi_{n,z}\rangle\langle\varphi_{n,z}|$ has kernel $\Pi_z(x,y) = \frac{1}{2\pi\hbar_b^2}e^{-\frac{|x-y|^2 - 2i(x^\perp\cdot y + 2z^\perp\cdot(x-y))}{4\hbar_b^2}}$

So $\nabla_z^\perp\Pi_z(x,y) = \frac{i}{\hbar_b^2}(x-y)\Pi_z(x,y)$ or $\boxed{\nabla_z^\perp\Pi_z = \frac{1}{i\hbar_b^2}[\Pi_z, X]}$ $(*)$

Semi-classical limit:

To a density matrix γ associate $m_\gamma(n,z) := \frac{1}{2\pi\hbar_b^2}\langle\varphi_{n,z}|\gamma\varphi_{n,z}\rangle$ (Hushimi measure)

Semi-classical density $\rho_{\gamma_b}^x(z) := \frac{1}{2\pi\hbar_b^2}\mathrm{Tr}[\gamma\Pi_z]$

Truncated semi-classical density $\rho_{\gamma_b}^{x,\le N}(z) := \sum_{n=0}^{N} m_\gamma(n,z)$

prop: convergence of $\rho_\gamma^{x,\le N}$

Let γ_b be a FDM s.t. $\mathrm{Tr}[\gamma_b\mathcal{L}_b] \le C$, then $\forall\varphi\in C_c^\infty(\mathbb{R}^2)$,

$$\left|\int_{\mathbb{R}^2}\varphi(\rho_{\gamma_b} - \rho_{\gamma_b}^{x,\le N})\right| \le C(\varphi)\left(\frac{1}{\sqrt{N}}\underbrace{\mathrm{Tr}[\gamma_b\Pi_{>N}\mathcal{L}_b]^{1/2}}_{\xrightarrow[N\to+\infty]{}0} + \underbrace{\sqrt{N}\hbar_b}_{\ll 1 \iff N \ll \frac{1}{\hbar_b}}\mathrm{Tr}[\gamma_b\mathcal{L}_b]^{1/2}\right)$$

prop: Gyrokinetic transport of $\rho_{\gamma_b}^{x,\le N}$

Let $t\in\mathbb{R}_+$, $\gamma_b(t)$ be a FDM, $W\in W^{4,\infty}(\mathbb{R}^2)$ assume

$\partial_t\gamma_b(t) = \frac{1}{i\hbar_b^2}\mathrm{Tr}[\mathcal{L}_b + W, \gamma_b(t)]$, $\mathrm{Tr}[\gamma_b(t)\mathcal{L}_b] \le C$

then there exists a choice of $1 \ll N \ll \frac{1}{\hbar_b^2}$ s.t.

$\forall\varphi\in C_c^\infty(\mathbb{R}^2)$, $\int_{\mathbb{R}^2}\varphi\left(\partial_t\rho_{\gamma_b(t)}^{x,\le N} + \nabla^\perp W\cdot\nabla\rho_{\gamma_b(t)}^{x,\le N}\right) \xrightarrow[b\to+\infty]{} 0$

Central computation:

$$\partial_t\rho_{\gamma_b}^x(z) = \frac{1}{2\pi\hbar_b^2}\mathrm{Tr}[\Pi_z\partial_t\gamma_b] \overset{\text{Hartree}}{=} \frac{1}{2\pi\hbar_b^2}\cdot\frac{1}{i\hbar_b^2}\mathrm{Tr}[\Pi_z[\mathcal{L}_b + W, \gamma_b]] \overset{\text{Tr cyclicity}}{=} \frac{1}{2i\pi\hbar_b^4}\mathrm{Tr}[\gamma_b[\Pi_z, \mathcal{L}_b + W]]$$

$$= \frac{1}{2i\pi\hbar_b^4}\mathrm{Tr}[\gamma_b[\Pi_z, W]]$$

$$\nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^x(z) = -\nabla W(z)\cdot\frac{1}{2\pi\hbar_b^2}\mathrm{Tr}[\gamma_b\nabla_z^\perp\Pi_z] \overset{(*)}{=} -\frac{1}{2i\pi\hbar_b^4}\nabla W(z)\cdot\mathrm{Tr}[\gamma_b[\Pi_z, X]]$$

So

$$\partial_t\rho_{\gamma_b}^x(z) + \nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{x,\le N}(z) = \frac{1}{2i\pi\hbar_b^4}\mathrm{Tr}\Big[\gamma_b\underbrace{[\Pi_z, W - \nabla W(z)\cdot X]}_{\Pi_z(x,y)(W(y) - W(x) - \nabla W(z)\cdot(y-x))}\Big]$$