Derivations of Gibbs measures for the focusing 1D nonlinear Schrödinger equation

Kick-off Meeting

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Outline of Talk

- Introduce and motivate Gibbs measures for the nonlinear Schrödinger equation (NLS)
- Discuss the construction of Gibbs measures for a certain nonlocal quintic nonlinear Schrödinger equation
- Define an analogue of Gibbs measures for many-body quantum mechanics
- Describe the convergence of these analogues as the number of particles goes to infinity

Hartree Equation

- Define: For $s \in \mathbb{R}$, we say $u \in H^s(\mathbb{T})$ if

$$(1+|k|^2)^{s/2}\hat{u}(k) \in \ell^2,$$

  $\|u\|_{H^s}^2 := \sum_{k\in\mathbb{Z}}(1+|k|^2)^s|\hat{u}(k)|^2$. Say u has **Sobolev regularity s**.
- We will consider the following form of the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t \varphi = (-\Delta+\kappa)\varphi + N^{(i)}(\varphi) \\ \varphi_0 \in H^s(\mathbb{T}), \end{cases}$$

  where we take $(x,t) \in \mathbb{T}\times\mathbb{R}$ and
  - $N^{(2)}(\varphi) := \int dy\, w(x-y)|\varphi(y)|^2\varphi(x) = (w * |\varphi|^2)\varphi(x)$
  - $N^{(3)}(\varphi) := \int dy\, dz\, w(x-y)w(y-z)w(z-x)|\varphi(y)|^2|\varphi(z)|^2\varphi(x) \neq (w*|\varphi|^2)^2\varphi(x)$
- $w : \mathbb{T} \to \mathbb{R}$ the **interaction potential** and $\kappa > 0$ a **chemical potential**
- This regime is called the **Hartree equation**
- If there are positivity assumptions on w, this called the **defocusing** problem. Otherwise we call it the **focusing** problem

- We have two **conserved** quantities associated with the Hartree equation:

$$M(\varphi) := \int dx\, |\varphi(x)|^2,$$

$$H(\varphi) := \underbrace{\int dx\, |\nabla\varphi(x)|^2 + \kappa|\varphi(x)|^2}_{H_0} + \mathcal{W}^{(i)},$$

  where
  - $\mathcal{W}^{(2)} := \frac{1}{2}\int dx\, dy\, |\varphi(x)|^2 w(x-y)|\varphi(y)|^2$
  - $\mathcal{W}^{(3)} := \frac{1}{3}\int dx\, dy\, dz\, w(x-y)w(y-z)w(z-y)|\varphi(y)|^2|\varphi(z)|^2|\varphi(x)|^2|\varphi(y)|^2$
- Idea: We use these conservation quantities to prove well-posedness
- At **low regularity**, we need a "substitute for conservation laws"

Gibbs measures

We can define the **Gibbs measure** associated to the NLS

$$d\mathbb{P}_{Gibbs} := \frac{1}{z} e^{-H(\varphi)} d\varphi,$$

which is supported on the space of initial conditions of the NLS. Here

- z is the **partition function**
- H is the **Hamiltonian**
- $d\varphi$ is the (formal) **infinite dimensional Lebesgue measure**

These measures were first studied in CQFT literature (summaries: Nelson ('73), Glimm-Jaffe ('81), Simon ('74)). To make the construction rigorous for a positive L^1 interaction potential w, one realises the Gibbs measure as a weighted **Wiener measure**

$$d\mathbb{P}_{Gibbs} = \frac{1}{z} e^{-\mathcal{W}} d\mu.$$

Gibbs Measures II

- Define the **flow map** S_t of the Hartree equation

$$\begin{cases} i\partial_t \varphi = (-\Delta + \kappa)\varphi + N^{(i)}(\varphi) \\ \varphi_0 \in H^s(\mathbb{T}), \end{cases}$$

  by $S_t\varphi_0(x) := \varphi(x,t)$
- A measure is *invariant under the flow* if $d\nu(A) = d\nu(S_{-t}A)$, where A is a measurable subset of the space of initial conditions

Theorem (Liouville's theorem)

For a **finite** *dimensional Hamiltonian system*

$$\begin{cases} \dot{p}_j = \frac{\partial H}{\partial q_j} \\ \dot{q}_j = -\frac{\partial H}{\partial p_j} \end{cases}$$

and a non-negative smooth function g, $g(H)d\mathrm{Leb}$ is invariant under the flow.

Gibbs Measures III

- We want to see what functions lie in this measures support
- Note $\int |\nabla \varphi|^2 = c \sum_{k\in\mathbb{Z}} |k|^2 |a_k|^2$, where $a_k := \hat{\varphi}(k)$
- Heuristically, for $\kappa = 0$

$$
\begin{aligned}
d\mu &\sim e^{-c\sum_{k\in\mathbb{Z}} |k|^2|a_k|^2} \prod_{k\in\mathbb{Z}} da_k \\
&= \prod_{k\in\mathbb{Z}} e^{-c|k|^2|a_k|^2} da_k
\end{aligned}
$$

- $|k|\hat{\varphi}(k)$ has a Gaussian distribution – gives rise to a **random Fourier series**
- Repeating for $\kappa \neq 0$, a function in the support of μ is given by

$$
\varphi^\omega(x) \equiv \varphi(x) = \sum_{k\in\mathbb{N}} \frac{g_k(\omega)}{\sqrt{|k|^2+\kappa}} e^{2\pi i k x},
$$

where g_k are **i.i.d. centred complex Gaussians**.

Gibbs Measures IV

- We want to know the **regularity** of functions in the support of μ.

$$\mathbb{E}_\mu\left[\|\varphi\|_{H^s}^2\right] = \mathbb{E}_\mu\left[\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\frac{|g_k(\omega)|^2}{|k|^2+\kappa}\right]$$
$$\sim \sum_{k\in\mathbb{Z}}(1+|k|^2)^{s-1}$$

- Finite if $2s-2<-1$, i.e. $s<1/2$.
- A function in the support of μ is in $H^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon>0$. Moreover $\mu(H^s)=1$ for any $s<1/2$ and $\mu(H^s)=0$ for any $s\geq 1/2$
- So for positive $w\in L^\infty$

$$\mathcal{W}\leq \frac{1}{2}\|w\|_{L^\infty}\|\varphi\|_{L^2}^4$$

so $\mathcal{W}\in[0,\infty)$ almost surely

Focusing potentials

- Bourgain ('94) and Zhidkov ('91) showed that the Gibbs measure is **invariant** under the flow of the NLS and Bourgain showed that the NLS is **almost surely globally well posed** for functions in the support of the Gibbs measure.
- Bourgain also showed that in the case of a **non-positive** interaction potential, one requires a **truncation** in the mass of the problem.
- Let $f \in C_0^\infty(\mathbb{R})$, and define

$$d\mathbb{P}^f_{Gibbs} := \frac{1}{z^f} e^{-\mathcal{W}} f\left(\|\varphi\|^2_{L^2}\right) d\mu$$

- In what follows, whenever we consider a Gibbs measure, we consider the **truncated measure**, and we drop the f dependence from our notation.
- This measure is **absolutely continuous** with respect to the Wiener measure for **local nonlocal Schrödinger equation**, so has the same support. Bourgain ('94) also showed that this measure is **invariant** and the NLS is globally well posed for any function in its support.
- This extends trivially to the nonlocal case for only the **cubic NLS**, and we have to work harder for the **quintic Hartree equation**.

Quintic Gibbs measure

$$N^{(3)}(\varphi) := \int dy\, dz\, w(x-y)w(y-z)w(z-x)|\varphi(y)|^2|\varphi(z)|^2\varphi(x)$$

Theorem (R-Sohinger '23)

Consider the Hartree equation with nonlinearity given by $N^{(3)}$ and $w \in L^{\frac{3}{2}}$. The following claims hold.

(i) *For $B > 0$ sufficiently small and φ given by the random Fourier series above, we have*

$$e^{-\frac{1}{3}\int dx\, dy\, dz\, w(x-y)\, w(y-z)\, w(x-z)\, |\varphi(x)|^2|\varphi(y)|^2|\varphi(z)|^2}\chi_{(M\leq B)} \in L^1(d\mu)\,.$$

In particular, for f with $\mathrm{supp}(f) \subset [-B, B]$, we get that the Gibbs measure is a well-defined probability measure.

(ii) *Consider $s \in (0, \frac{1}{2})$. The measure $\mathbb{P}_{\mathrm{Gibbs}}$ is invariant under the flow of quintic Hartree equation. Furthermore, the quintic Hartree equation admits global solutions for $\mathbb{P}_{\mathrm{Gibbs}}$-almost every $u_0 \in H^s(\Lambda)$.*

Ideas of proof

- We show local well-posedness for $w \in L^{\frac{3}{2}}$ – uses $X^{s,b}$ spaces and multilinear estimates (Fourier analysis)
- We use Sobolev embedding to notice

$$\left|N^{(3)}\right| \leq \frac{1}{3}\|w\|_{L^{\frac{3}{2}}}^{3}\|\varphi\|_{H^{\frac{1}{3}}}^{6}$$

- We argue analogously to Bourgain ('94) to get normalisability
- Following Bourgain, we use a Galerkin approximation to obtain invariance of the measure.

Many-body quantum mechanics

- We restrict ourselves to the **two-body** case
- We define $L^2_s(\mathbb{T}^N)$ to as the symmetric subspace of $L^2(\mathbb{T}^N)$. I.e.

$$u(x_1, \ldots, x_N) = u(x_{\pi 1}, \ldots, u_{\pi N})$$

  for any permutation $\pi \in S_N$.
- This framework corresponds to studying **bosons**.
- We consider the *N-body Hamiltonian* on $L^2_s(\mathbb{T}^N)$

$$H_N := \sum_{j=1}^{N} (-\Delta_j + \kappa) + \frac{1}{N} \sum_{i<j}^{N} w(x_i - x_j)$$

Connecting Hartree and many-body problems

- We can view the Hartree equation as the **large particle limit** of

$$i\partial_t \Psi_N = H_N \Psi_N$$

- If the initial condition is "close" to being factorised: $\Psi_N(x,0) \sim \varphi_0^{\otimes n}$, then

$$\Psi_N(x,t) \sim \varphi^{\otimes n}(x,t),$$

  where φ satisfies

$$\begin{cases} i\partial_t \varphi = (-\Delta + \kappa)\varphi + (w * |\varphi|^2)(x)\varphi(x) \\ \varphi_0. \end{cases}$$

- This is made rigorous using **reduced density matrix** – Hepp ('74), Ginibre-Velo ('79), Spohn ('80).

Gibbs measures in many-body quantum mechanics

- **Question:** What do Gibbs measures correspond to in many body quantum mechanics?
- **Idea I:** Fix the number of particles – **canonical ensemble**
- Consider operator $P_\tau^{(N)} := \frac{1}{Z_N} e^{-\frac{1}{\tau} H_N}$
- Expanding spectrally:

$$P_\tau^{(N)} = \frac{1}{\sum_{k \in \mathbb{Z}} e^{-\lambda_{N,k}/\tau}} \sum_{k \in \mathbb{Z}} e^{-\lambda_{N,k}/\tau} u_k u_k^*$$

- Taking $\tau \to \infty$, we get a **uniform distribution** onto the spectral projections
- Taking $\tau \to 0$, we get a δ measure on the **ground state projection**
- To get a non-trivial limit, we need to vary the number of particles – **grand canonical ensemble**. We also need to vary temperature with the number of particles $\tau = N$.

Grand canonical ensemble

- We work with the *bosonic Fock space*

$$\mathcal{F} := \bigoplus_{p \geq 0} L^2_s(\mathbb{T}^p)$$

- For $g \in L^2(\mathbb{T})$, we consider the *creation and annihilation* operators

$$\left(b^*(g)\Psi\right)^{(p)}(x_1, \ldots, x_p) := \frac{1}{\sqrt{p}} \sum_{i=1}^{p} g(x_i) \Psi^{(p-1)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p)\,,$$

$$\left(b(g)\Psi\right)^{(p)}(x_1, \ldots, x_p) := \sqrt{p+1} \int dx\, \overline{g(x)} \Psi^{(p+1)}(x, x_1, \ldots, x_p)$$

- These are operator-valued distributions – “test” against a function to get an operator
- These satisfy the **canonical commutation relations**

$$[b(g_1), b^*(g_2)] = \langle g_1, g_2 \rangle_{\mathfrak{h}}\,, \quad [b(g_1), b(g_2)] = [b^*(g_1), b^*(g_2)] = 0\,,$$

  for all $g_1, g_2 \in L^2(\mathbb{T})$.
- We consider the *rescaled creation and annihilation operators* $\varphi^*_\tau(g) := \tau^{-1/2} b^*(f)$ and $\varphi_\tau(g) := \tau^{-1/2} b(f)$.

Grand Canonical ensemble II

- For a closed linear operator ξ on $L^2_s(\mathbb{T}^N)$, we identify it with its Schwartz kernel, $\xi(x_1, \dots, x_p, y_1, \dots, y_p)$.
- We define the **lift** of an operator ξ to Fock space as

$$\Theta_\tau(\xi) := \int dx_1 \dots dx_p \, dy_1 \dots dy_p \, \xi(x_1, \dots, x_p, y_1, \dots, y_p) \\ \varphi^*_\tau(x_1) \dots \varphi^*_\tau(x_p) \varphi_\tau(y_1) \dots \varphi_\tau(y_p),$$

  where $\varphi^*_\tau(x) := \varphi^*_\tau(\delta_x)$ is the **distribution kernel** of φ^*_τ.
- We can interpret φ^*_τ and φ_τ as the **quantum analogues** of $\overline{\varphi}$ and φ. In particular, the canonical commutation relations imply

$$[\varphi_\tau(x), \varphi^*_\tau(y)] = \frac{1}{\tau}\delta(x - y)$$

- For $h = -\Delta + \kappa$ and $W^{(2)}$ the two particle operator which acts by multiplication by $w(x_1 - x_2)$, define

$$H_{\tau,0} := \Theta_\tau(h), \quad \mathcal{W}^{(2)}_\tau := \frac{1}{2}\Theta_\tau(W^{(2)}).$$

Quantum State

- The **interacting quantum Hamiltonian** is defined as

$$H_\tau := H_{\tau,0} + \mathcal{W}_\tau,$$

  and the **rescaled particle number** as

$$\mathcal{N}_\tau := \int dx\, \varphi_\tau^*(x)\varphi_\tau(x).$$

- We define the **grand canonical ensemble** as $P_\tau := e^{-H_\tau}$.
- For $\mathcal{A} : \mathcal{F} \to \mathcal{F}$, the **quantum state** $\rho_\tau^f(\cdot) \equiv \rho_\tau(\cdot)$ is defined as

$$\rho_\tau(\mathcal{A}) := \frac{\mathrm{Tr}_{\mathcal{F}}\left(\mathcal{A}P_\tau f\left(\mathcal{N}_\tau\right)\right)}{\mathrm{Tr}_{\mathcal{F}}\left(P_\tau f\left(\mathcal{N}_\tau\right)\right)}.$$

- We consider

$$Z_\tau := \mathrm{Tr}\left(e^{-H_\tau} f\left(\mathcal{N}_\tau\right)\right), \quad Z_{\tau,0} := \mathrm{Tr}\left(e^{-H_{\tau,0}}\right), \quad \mathcal{Z}_\tau := \frac{Z_\tau}{Z_{\tau,0}},$$

  the quantum, free quantum, and relative quantum partition functions.

Classical State

- In analogy to the quantum case, for a bounded operator on $\mathfrak{h}$, we define the random variable

$$\Theta(\xi) := \int dx_1 \dots dx_p \, dy_1 \, \dots dy_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \times \overline{\varphi}(x_1) \dots \overline{\varphi}(x_p) \varphi(y_1) \dots \varphi(y_p),$$

and

$$H := \underbrace{\Theta(h)}_{H_0} + \underbrace{\frac{1}{2}\Theta(W)}_{\mathcal{W}^{(2)}}$$

- We define the **classical state**, $\rho = \rho^f$ of a random variable X as

$$\rho(X) = \frac{\int d\mu \, X e^{-\mathcal{W}^{(2)}} f\left(\|\varphi\|_{L^2}^2\right)}{\int d\mu \, e^{-\mathcal{W}^{(2)}} f\left(\|\varphi\|_{L^2}^2\right)}.$$

- We consider the **classical partition function**

$$z := \int d\mu \, e^{-\mathcal{W}^{(2)}} f\left(\|\varphi\|_{L^2}^2\right).$$

Correlation functions

- Both the classical and quantum states can be characterised through their **correlation functions**, which have kernels defined as

$$\gamma_p(x_1,\dots,x_p;y_1,\dots,y_p) := \rho\left(\overline{\varphi}(y_1)\dots\overline{\varphi}(y_p)\varphi(x_1)\dots\varphi(x_p)\right),$$
$$\gamma_{\tau,p}(x_1,\dots,x_p;y_1,\dots,y_p) := \rho_\tau(\varphi_\tau^*(y_1)\dots\varphi_\tau^*(y_p)\varphi_\tau(x_1)\dots\varphi_\tau(x_p)).$$

Theorem (R-Sohinger '23)

Let $w \in L^\infty$ be real valued and even. Given $p \in \{1,2,\dots\}$, we have

$$\lim_{\tau\to\infty} \mathrm{Tr}_{L^2(\mathbb{T}^p)} |\gamma_{\tau,p} - \gamma_p| = 0.$$

Moreover

$$\lim_{\tau\to\infty} \mathcal{Z}_\tau = z.$$

- We also prove similar results for $w \in L^1$ and $w = -\delta$

Quintic Result

- We take

$$\mathcal{W}_\tau^{(3)} := \frac{1}{3}\Theta_\tau(W^{(3)}),$$

where $W^{(3)}$ acts as multiplication by $w(x_1 - x_2)w(x_2 - x_3)w(x_3 - x_1)$.

$$H_\tau^{(n)} = \frac{1}{\tau}\sum_{j=1}^{n} -\Delta_j + \frac{1}{3\tau^2}\sum_{i\neq j\neq k\neq i}^{n} w(x_1 - x_2)w(x_2 - x_3)w(x_3 - x_1)$$

$$\mathcal{W}^{(3)} := \frac{1}{3}\Theta(W^{(3)}).$$

Theorem (R-Sohinger '22)

Let $w \in L^\infty$ be real valued and even. Given $p \in \{1, 2, \ldots\}$, we have

$$\lim_{\tau\to\infty} \operatorname{Tr}_{L^2(\mathbb{T}^p)} |\gamma_{\tau,p} - \gamma_p| = 0.$$

Moreover

$$\lim_{\tau\to\infty} \mathcal{Z}_\tau = z.$$

- **Key Difference:** We can only extend our results to $L^{\frac{3}{2}}$ potentials

Relation to Previous Derivations of Gibbs Measures

- The problem has been well studied in the case of a **positive** interaction potential

1. **Lewin-Nam-Rougerie ('15)**: 1D results using variational method. Non-tranlation invariant interaction for $d = 2, 3$.
2. **Fröhlich-Knowles-Schlein-Sohinger ('17)** Bounded potentials in $d = 2, 3$ with modified Gibbs state. New proof of $d = 1$.
3. **LNR ('18)**: $1D$ non-periodic subharmonic trapping potential.
4. **LNR ('18)**: $2D$ smooth interaction without modified Gibbs state.
5. **FKSS ('18)**: Time dependent problem for $1D$.
6. **Sohinger ('19)**: Results of FKSS ('17) extended to optimal $w \in L^q$.
7. **FKSS ('20)**: Results of FKSS ('17) for $d = 2, 3$ without modified Gibbs state.
8. **LNR ('20)**: Extension to $d = 3$.
9. **FKSS ('22)**: ϕ_2^4 Euclidean field theory for potential with contracting range.

- Ours is the first known result for an interaction potential **without positivity assumption** in the two-body case. It is the first known result in the **three-body** case.

Results for three-body interactions

- **T.Chen-Pavlovic ('10)**: Shows quintic NLS is an effective equation for a Gross-Pitaevskii range potential. **Xie ('10)**: Similar results for p-body interactions .
- **X.Chen ('12)** Shows a nonlocal NLS is an effective equation for a different three-body interaction.
- **Lee ('20)** Rate of convergence for a particular three-body interaction similar to X.Chen.
- BEC studied by **Nam-Ricaud-Triay '22,'22**

Bounded potential proof

- $\rho_\tau(\mathcal{A}) = \frac{\tilde{\rho}_{\tau,1}(\mathcal{A})}{\tilde{\rho}_{\tau,1}(1)}$, where $\tilde{\rho}_{\tau,z}(\mathcal{A}) := \frac{\mathrm{Tr}(\mathcal{A}\mathrm{e}^{-\mathrm{H}_{\tau,0}+\mathrm{z}\mathcal{W}_\tau})}{Z_{\tau,0}}$.
- We expand $\rho_\tau(\Theta(\xi))$ and $\rho(\Theta(\xi))$ as power series using a **Duhamel expansion**: $e^{X+\zeta Y} = e^X + \zeta\int_0^1 dt\, e^{(1-t)X} Y e^{t(X+\zeta Y)}$

$$A_\tau^\xi(\zeta) := \sum_{m=0}^{M-1} a_{\tau,m}^\xi \zeta^m + R_{\tau,M}^\xi(\zeta), \quad A^\xi(\zeta) := \sum_{m=0}^{M-1} a_m^\xi \zeta^m + R_M^\xi(\zeta),$$

where

$$a_{\tau,m}^\xi := \frac{(-1)^m}{Z_{\tau,0}} \mathrm{Tr}\bigg(\int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{m-1}} dt_m\, \Theta_\tau(\xi) e^{-(1-t_1)H_{\tau,0}} \mathcal{W}_\tau$$
$$\times\, e^{-(t_1-t_2)H_{\tau,0}} \mathcal{W}_\tau e^{-(t_2-t_3)H_{\tau,0}} \dots e^{-(t_{m-1}-t_m)H_{\tau,0}} \mathcal{W}_\tau e^{-t_m H_{\tau,0}} f(\mathcal{N}_\tau) \bigg),$$

$$R_{\tau,M}^\xi(\zeta) := \frac{(-\zeta)^M}{Z_{\tau,0}} \mathrm{Tr}\bigg(\int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{M-1}} dt_M\, \Theta_\tau(\xi) e^{-(1-t_1)H_{\tau,0}}$$
$$\times\, \mathcal{W}_\tau e^{-(t_1-t_2)H_{\tau,0}} \dots\dots e^{-(t_{M-1}-t_M)H_{\tau,0}} \mathcal{W}_\tau e^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f(\mathcal{N}_\tau) \bigg).$$

Bounded potential proof II

- Need to prove
  1. The explicit terms satisfy sufficient **bounds**
  2. We can **rewrite** the remainder terms using the explicit terms
  3. We get **convergence** of the quantum explicit terms to the classical ones

Bounds on the explicit terms

$$a^{\xi}_{\tau,m} := \frac{(-1)^m}{Z_{\tau,0}} \mathrm{Tr}\Bigg(\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m \, \Theta_\tau(\xi) e^{-(1-t_1)H_{\tau,0}} \mathcal{W}_\tau$$

$$\times \, e^{-(t_1-t_2)H_{\tau,0}} \mathcal{W}_\tau e^{-(t_2-t_3)H_{\tau,0}} \ldots e^{-(t_{m-1}-t_m)H_{\tau,0}} \mathcal{W}_\tau e^{-t_m H_{\tau,0}} f\left(\mathcal{N}_\tau\right) \Bigg).$$

- Notice: $\mathcal{W}_\tau$ **is not in the exponential**
- Notice: Sum of the times in the exponential is 1
- **Both $\Theta_\tau(\xi)$ and $\mathcal{W}_\tau$ grow like $\mathcal{N}_\tau$**
- Idea: Apply Hölder's inequality. Sum of exponents is 1, giving **supremum norms on the $\Theta_\tau(\xi)$ and $\mathcal{W}_\tau$ terms** and a $\mathrm{Tr}(e^{-H_{\tau,0}}) =: Z_{\tau,0}$
- $\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m = \frac{1}{m!}$

Remainder term

$$R^{\xi}_{\tau,M}(\zeta) := \frac{(-\zeta)^M}{Z_{\tau,0}} \mathrm{Tr}\bigg(\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{M-1}} dt_M \, \Theta_\tau(\xi) e^{-(1-t_1)H_{\tau,0}}$$
$$\times \mathcal{W}_\tau e^{-(t_1-t_2)H_{\tau,0}} \ldots \ldots e^{-(t_{M-1}-t_M)H_{\tau,0}} \mathcal{W}_\tau e^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f\left(\mathcal{N}_\tau\right) \bigg).$$

- Feynman-Kac: $e^{t(\Delta-V)}(x;\tilde{x}) = \int \mathbb{W}^t_{x,\tilde{x}}(d\omega) e^{-\int_0^t ds\, V(\omega(s))}$
- We use the **Feynman-Kac formula** to rewrite remainder term using the explicit term
- Feynman-Kac gives us a formula for $e^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)}$
- We thus get analytic power series for ρ_τ and ρ

- Key to getting convergence is proving bounds on the **untruncated explicit terms**

$$a^{\xi}_{\tau,m} := \frac{(-1)^m}{Z_{\tau,0}} \mathrm{Tr}\bigg(\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m \, \Theta_\tau(\xi) e^{-(1-t_1)H_{\tau,0}} \mathcal{W}_\tau \\ \times e^{-(t_1-t_2)H_{\tau,0}} \mathcal{W}_\tau e^{-(t_2-t_3)H_{\tau,0}} \ldots e^{-(t_{m-1}-t_m)H_{\tau,0}} \mathcal{W}_\tau e^{-t_m H_{\tau,0}} \bigg).$$

- By rearranging, we need to estimate a something of the form of

$$\frac{1}{Z_{\tau,0}} \int dx_1 \ldots dx_{m+p} \, dy_1 \ldots dy_{m+p} \, \mathrm{Tr}\bigg(\prod_{i=1}^{m} (\varphi^*_\tau(x_i) \varphi^*_\tau(y_i) \varphi_\tau(x_i) \varphi_\tau(y_i)) \\ \times \prod_{i=1}^{p} \varphi^*_\tau(x_{m+i}) \prod_{i=1}^{p} \varphi_\tau(y_{m+i}) e^{-H_{\tau,0}} \bigg)$$

- **Idea:** φ^*_τ and φ_τ heuristically act like $\overline{\varphi}$ and φ respectively, and the **trace** is heuristically like an expectation $\longrightarrow$ **Wick's theorem**

Wick's theorem

Theorem (Classical Wick's theorem)

Given suitable g*, we let* $\varphi(g) := \langle g, \varphi \rangle$ *and* $\overline{\varphi}(g) := \langle \varphi, g \rangle$*. Furthermore, we let* $(\varphi)^*(g)$ *denote either* $\varphi(g)$ *or* $\overline{\varphi}(g)$*. Then, given* $N \geq 0$ *and* $g_1, \ldots, g_N \in H^{-\frac{1}{2}+\varepsilon}$*, we have*

$$\mathbb{E}_\mu \left[\prod_{i=1}^{N} (\varphi(g_i))^{*_i} \right] = \sum_{\Pi \in \mathcal{M}(n)} \prod_{(i,j) \in \Pi} \mathbb{E}_\mu \left[(\varphi(g_i))^{*_i} (\varphi(g_j))^{*_j} \right],$$

where the sum is taken over all complete pairings of $\{1, \ldots, n\}$*, and where edges of* Π *are denoted by* (i, j) *with* $i < j$*.*

- Since φ has a random Fourier series consisting only of **Gaussians**, we only recover the "diagonal" terms in the product.

Wick's theorem in the quantum setting

- A quantum analogue of Wick's theorem allows us to look at pairings of the following **graphs**

Time Dependent Problem

- We introduce time evolution

$$\Psi^t\Theta(\xi) := \int dx_1 \dots dx_p \, dy_1 \dots dy_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \\ \times \overline{S_t\varphi}(x_1) \dots \overline{S_t\varphi}(x_p) S_t\varphi(y_1) \dots S_t\varphi(y_p) \,,$$

$$\Psi^t_\tau \mathcal{A} := e^{it\tau H_\tau} \mathcal{A} \, e^{-it\tau H_\tau} \,.$$

Theorem (R-Sohinger '22,'23)

Let w be bounded and even. Let $m \in \{1, 2, \dots\}$, $p_1, \dots, p_m \in \{1, 2, \dots\}$, $\xi_1 \in \mathcal{L}(\mathfrak{h}^{(p_1)}), \dots, \xi_m \in \mathcal{L}(\mathfrak{h}^{(p_m)})$, and $t_1, \dots, t_m \in \mathbb{R}$ be given. Then

$$\lim_{\tau \to \infty} \rho^\varepsilon_\tau(\Psi^{t_1}_\tau \Theta_\tau(\xi_1) \dots \Psi^{t_m}_\tau \Theta_\tau(\xi_m)) = \rho(\Psi^{t_1}\Theta(\xi_1) \dots \Psi^{t_m}\Theta(\xi_m)) \,.$$

- Use **Schwinger-Dyson expansion** in bounded case. Expand to unbounded using a **diagonal argument**